THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Tutorial 5 (March 22)

The following were discussed in the tutorial this week.

Exercise 1. Let $E \subseteq$ be a measurable set and let $f : E \to \mathbb{R}$ be a measurable function. Suppose $f'(x)$ exists for all $x \in E$. Show that

$$m^*(f(E)) \leq \int_E |f'|.$$

Definition 1. Let $f : [a, b] \to \mathbb{R}$, and let $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathrm{par}[a, b]$. Define

$$t(f; \pi) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

and the **total variation** of f over $[a, b]$ by

$$T_a^b(f) := \sup\{t(f; \pi) : \pi \in \mathrm{par}[a, b]\}.$$

We also set $T_a^a(f) := 0$. Similarly, we define the **positive** and **negative variation** $P_a^b(f), N_a^b(f)$ by replacing $|\cdot|$ with $[\,\cdot\,]^+$ and $[\,\cdot\,]^-$, respectively.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$. Then*

(a) $T_a^b(f) = T_a^c(f) + T_c^b(f)$ for any $c \in [a, b]$;

(b) $T_a^x(f)$ is non-negative and increasing for $x \in [a, b]$.

Corresponding results hold for $P_a^x(f)$ and $N_a^x(f)$.

Definition 2. We say that $f : [a, b] \to \mathbb{R}$ is of **bounded variation** over $[a, b]$ if $T_a^b(f) < +\infty$. In symbol, we write $f \in \mathrm{BV}[a, b]$.

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$. Then*

(a) $T_a^b(f) = P_a^b(f) + N_a^b(f)$;

(b) $f(b) - f(a) = P_a^b(f) - N_a^b(f)$ if $f \in \mathrm{BV}[a, b]$.

Theorem 3 (Jordan decomposition for BV functions)**.** *$f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there is a pair of increasing functions $g, h : [a, b] \to \mathbb{R}$ such that $f = g - h$. Furthermore, if $g(b) - g(a) + h(b) - h(a) = T_a^b(f)$, then*

$$g(x) - g(a) = P_a^x(f) \quad \text{and} \quad h(x) - h(a) = N_a^x(f).$$

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be monotone increasing. Then*

(a) $f'(x)$ exists a.e.;

(b) f' is measurable;

(c) $f' \in \mathcal{L}[a, b]$ with $0 \le \int_a^b f' \le f(b) - f(a)$.

Remark. The last inequality can be strict: for example, the Cantor function.

Corollary 5. *If $f \in \mathrm{BV}[a, b]$, then $f'(x)$ exists a.e. and $f' \in \mathcal{L}[a, b]$.*

Exercise 2. Let $f \in \mathrm{BV}[a, b]$. Show that $\int_a^b |f'| \le T_a^b(f)$.

Definition 3. $f : [a, b] \to \mathbb{R}$ is said to be **absolutely continuous** on $[a, b]$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \varepsilon$ whenever $\{(x_i, x_i')\}_{i=1}^n$ is a finite collection of **non-overlapping** intervals such that $\sum_{i=1}^{n} |x_i - x_i'| < \delta$. In this case, we write $f \in \mathrm{ABC}[a, b]$.

Proposition 6. *Let $f \in \mathrm{ABC}[a, b]$. Then*

(a) f is continuous;

(b) f is of bounded variation;

(c) f has the Luzin N property: f maps sets of measure zero to sets of measure zero.

Remark. In fact, a function satisfying all three conditions is absolutely continuous. This is Banach-Zarecki Theorem.