Suggested Solution of Assignment 5(i)

1. From the definition of $m^*(A)$, show that

$$\begin{aligned} m^*(A) &= \inf\{m(G) : \text{ open } G \supseteq A\} \\ &= \inf\{m(G) : \text{ open } G \text{ and } G_0 \supseteq G \supseteq A\}, \end{aligned}$$

where G_0 is an open set containing A.

Solution. By taking $G_0 = \mathbb{R}$, it suffices to show the second equality. Moreover, by the monotonicity of m^* and the measurability of open sets, "$\leq$" clearly holds. It remains to prove "$\geq$". If $m^*(A) = +\infty$, then we are done. Suppose $m^*(A) < +\infty$. Let $\varepsilon > 0$. By the definition of $m^*(A)$, there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$m^*(A) + \varepsilon \geq \sum_{n=1}^{\infty} \ell(I_n).$$

Let $G := G_0 \cap \bigcup_{n=1}^{\infty} I_n$, which is open and satisfies $A \subseteq G \subseteq G_0$. Since each I_n is measurable with $\ell(I_n) = m(I_n)$, we have

$$m(G) \leq m(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon.$$

Hence $\inf\{m(G) : \text{ open } G \text{ and } G_0 \supseteq G \supseteq A\} \leq m^*(A) + \varepsilon$. Since ε is arbitrary, the result follows.

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2. Let $E \subseteq G$. Show that, $\forall\, U \subseteq \mathbb{R}$,

$$E \setminus U = E \setminus (U \cap G)$$

and

$$m^*(E \triangle (U \cap G)) \leq m^*(E \triangle U).$$

Solution. Since $E \subseteq G$, we have

$$E \setminus (U \cap G) = (E \setminus U) \cup (E \setminus G) = (E \setminus U) \cup \emptyset = E \setminus U,$$

so that

$$E \triangle (U \cap G) = [E \setminus (U \cap G)] \cup [(U \cap G) \setminus E] \subseteq (E \setminus U) \cup (U \setminus E) = E \triangle U.$$

Hence, by the monotonicity of m^*,

$$m^*(E \triangle (U \cap G)) \leq m^*(E \triangle U).$$

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3. Let $\mathcal{M} \ni E \subseteq (a,b) \subseteq \mathbb{R}$ and $\varepsilon > 0$. Show that $\exists$ disjoint open intervals $I_1, I_2, \ldots, I_n$ contained in (a,b) such that

$$m(E \triangle \bigcup_{i=1}^{n} I_i) < \varepsilon,$$

in two methods:

(a) Using Q1.

(b) Using Q2 (and (i) $\implies$ (iv) of the 1st principle of Littlewood for $m^*(E) < +\infty$).

Solution. (a) By Q1, there exists an open set G such that $E \subseteq G \subseteq (a,b)$ and $m(G) < m(E) + \varepsilon/2$. Since $m(G), m(E) < +\infty$, we have

$$m(G \setminus E) = m(G) - m(E) < \varepsilon/2.$$

As an open set, we can express $G = \bigcup_{i=1}^{\infty} I_i$, a countable disjoint union of open intervals. In particular, each $I_i \subseteq G \subseteq (a,b)$. Since

$$\sum_{i=1}^{\infty} m(I_i) = m(G) < +\infty,$$

we can find $n \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} m(I_i) < \varepsilon/2$. Now

$$m(\bigcup_{i=1}^{n} I_i \setminus E) \leq m(G \setminus E) < \varepsilon/2,$$

and

$$m(E \setminus \bigcup_{i=1}^{n} I_i) \leq m(G \setminus \bigcup_{i=1}^{n} I_i) = \sum_{n=n+1}^{\infty} m(I_i) < \varepsilon/2.$$

Therefore

$$m(E \triangle \bigcup_{i=1}^{n} I_i) = m(E \setminus \bigcup_{i=1}^{n} I_i) + m(\bigcup_{i=1}^{n} I_i \setminus E) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(b) By the 1st principle of Littlewood for $m^*(E) < +\infty$, there exist disjoint open intervals $I_1, I_2, \ldots, I_n$ such that

$$m(E \triangle \bigcup_{i=1}^{n} I_i) < \varepsilon.$$

Since the intersection of two open intervals is still an open interval (though possibly empty), $J_i := I_i \cap (a,b)$, $1 \leq i \leq n$, are disjoint open intervals contained in (a,b) such that

$$m(E \triangle \bigcup_{i=1}^{m} J_i) = m(E \triangle ((a,b) \cap \bigcup_{i=1}^{n} I_i)) \leq m(E \triangle \bigcup_{i=1}^{n} I_i) < \varepsilon,$$

where Q2 is used in the second inequality.

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