

4050 Hw3

1. Show that $\inf X \geq \inf Y$ whenever $X \subseteq Y (\subseteq \mathbb{R})$ and hence that $m^*(A) \uparrow$ (i.e. $m^*(A) \leq m^*(B)$ if $A \subseteq B (\subseteq \mathbb{R})$).

2. Let \mathcal{A} be an algebra of subsets of X . Show that \mathcal{A} is a σ -algebra if (and only if) \mathcal{A} is stable w.r.t. countable disjoint unions:

$$\bigcup_{n=1}^{\infty} A_n \text{ whenever } A_n \in \mathcal{A} \forall n \in \mathbb{N} \text{ and } A_m \cap A_n = \emptyset \forall m \neq n.$$

3. Suppose $[a, b] (\subseteq \mathbb{R})$ is covered by a finite family \mathcal{C} of open intervals. Show that $b-a \leq \text{sum of lengths of intervals in } \mathcal{C}$ (by MI to $n := \#(\mathcal{C})$, the number of elements of \mathcal{C}).

4. (cf. Royden 3rd, P52, Q51). Upper/Lower Envelopes of $f: [a, b] \rightarrow \mathbb{R}$. Define $h, g: [a, b] \rightarrow [-\infty, \infty]$ by, $\forall y \in [a, b]$,

$$h(y) := \inf \{ h_\delta(y) : \delta > 0 \}, \text{ where } h_\delta(y) := \sup \{ f(x) : x \in [a, b], |x-y| < \delta \}$$

$$g(y) := \sup \{ g_\delta(y) : \delta > 0 \}, \text{ where } g_\delta(y) := \inf \{ f(x) : x \in [a, b], |x-y| < \delta \}$$

Prove the following:

a. $g \leq f \leq h$ pointwisely on $[a, b]$, and $\forall x \in [a, b]$,
 $g(x) = f(x)$ iff f is lsc at x ($f(x) = h(x)$ iff f is usc at x)
 so $g(x) = h(x)$ iff f is continuous at x .

b. If f is bounded (so g, h are real-valued) then g is lsc & h is usc.

c. If φ is a lsc $\text{on } [a, b]$ such that $\varphi \leq f$ (pointwise) on $[a, b]$ then $\varphi \leq g$. State and show the corresponding result for h .

d. Let $C_n := \{ x \in [a, b] : h(x) - g(x) < \frac{1}{n} \} \forall n \in \mathbb{N}$. Then $C := \bigcap_{n=1}^{\infty} C_n$ is exactly the set of all continuity points of f and is a G_δ -set.

Note. More suggestive notations for g, h are f and \bar{f} .

5. Let $f: [a, b] \rightarrow [m, M]$. For each $P \in \text{par } [a, b]$, let $u(f; P)$ and $U(f; P)$ denote the lower/upper Riemann-sum functions. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1} \forall n$ and $\|P_n\| \rightarrow 0$ (the max of subinterval-lengths of P_n). Show that, $\forall x \in [a, b] \setminus A$
 $\lim_n (u(f; P_n))(x) = \underline{f}(x)$ and $\lim_n (U(f; P_n))(x) = \bar{f}(x)$, where A denotes the union of all end-points of $P_n \forall n \in \mathbb{N}$.