MATH 2230 Tutorial 5

1. Find and sketch the analytic domain of the following function:

$$f(z) = \frac{\mathrm{Log}(1+z^2)}{z-2}$$

Ans: 1°, $z-2 \neq 0 \Rightarrow z \neq 2$.

2°, since $\mathrm{Log}\, w$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$

$\therefore$ $\mathrm{Log}(1+z^2)$ is analytic at z

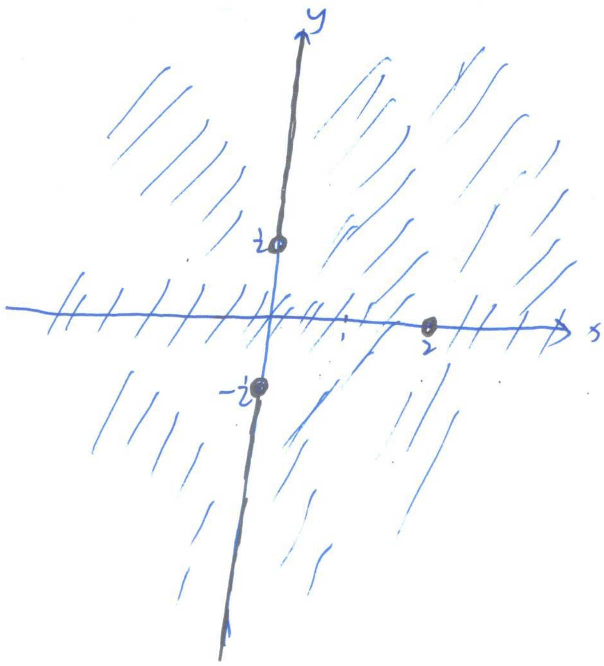
$\iff 1+z^2 \notin (-\infty, 0]$

$\iff z^2 \notin (-\infty, -1]$

$\iff z \notin \{ti \mid t \in [1, \infty) \text{ or } t \in (-\infty, -1]\}$

$\therefore$ the analytic domain of $f(z)$ is

$$\mathbb{C} \setminus (\{2\} \cup \{ti \mid t \in (-\infty, -1] \cup [1, +\infty)\})$$



2、Find all solutions for

$$e^{(1+i)\log z} = 4$$

Ans: write $z = re^{i\theta}$

$$e^{(1+i)\log z} = e^{(1+i)(\lg r + i(\theta+2n\pi))}$$

$$= e^{\lg r-(\theta+2n\pi)} e^{i(\lg r+\theta+2n\pi)}$$

$$= e^{\lg r-(\theta+2n\pi)} e^{i(\lg r+\theta)}$$

$\because 4 = e^{\lg 4}$

$\therefore \begin{cases} \lg r-(\theta+2n\pi) = 2\lg 2 & \text{for } n\in \mathbb{Z} \\ \lg r+\theta = 2m\pi & \text{for } m\in \mathbb{Z} \end{cases}$

$\therefore$ ~~$\lg r = \lg 2 + k\pi$ for $k \in \mathbb{Z}$~~

$\begin{cases} \lg r = \lg 2 + (m+n)\pi \\ \theta = (m-n)\pi - \lg 2 \end{cases}$ for $\forall\, m,n \in \mathbb{Z}$

3. Find $\cos^{-1} i$

Ans: $\cos^{-1} i = -i \log(i + i\sqrt{1-i^2})$

$= -i \log(i \pm \sqrt{2} i)$

$= -i \log(1 \pm \sqrt{2}) i$

$= -i \log(1+\sqrt{2}) e^{i\frac{\pi}{2}}$ or $-i \log(\sqrt{2}-1) e^{-i\frac{\pi}{2}}$

$= -i\left[\log(1+\sqrt{2}) + i(\frac{\pi}{2}+2n\pi)\right]$ or $-i\left[\log(\sqrt{2}-1) + i(-\frac{\pi}{2}+2n\pi)\right]$

$= (\frac{\pi}{2}+2n\pi) - i\log(1+\sqrt{2})$ or $(-\frac{\pi}{2}+2n\pi) - i\log(\sqrt{2}-1)$

4. Find $\tanh^{-1} 0$

Ans: $\tanh^{-1} 0$

$= \frac{1}{2} \log \frac{1+0}{1-0}$

$= \frac{1}{2} \log 1$

$= \frac{1}{2}\left[\lg 1 + i(2n\pi + 0)\right]$

$= n\pi i$

5. Find the value of

$$\text{Log}\left((1-i)^{i+2}\right)$$

Ans: $(1-i)^{i+2} = e^{(i+2)\text{Log}(1-i)}$

$$= e^{(i+2)\text{Log}\left[\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\right]}$$

$$= e^{(i+2)\text{Log}\sqrt{2}\,e^{-i\frac{\pi}{4}}}$$

$$= e^{(i+2)\left[\frac{1}{2}\lg 2 + i\left(-\frac{\pi}{4}+2n\pi\right)\right]}$$

$$= e^{\lg 2+\frac{\pi}{4}-2n\pi} \cdot e^{i\left(\frac{1}{2}\lg 2-\frac{\pi}{2}+4n\pi\right)}$$

$\therefore$ $\text{Log}\left[(1-i)^{i+2}\right]$

$$= \lg 2+\frac{\pi}{4}-2n\pi + i\left(\frac{1}{2}\lg 2-\frac{\pi}{2}+2m\pi\right), \quad n, m \in \mathbb{Z}$$