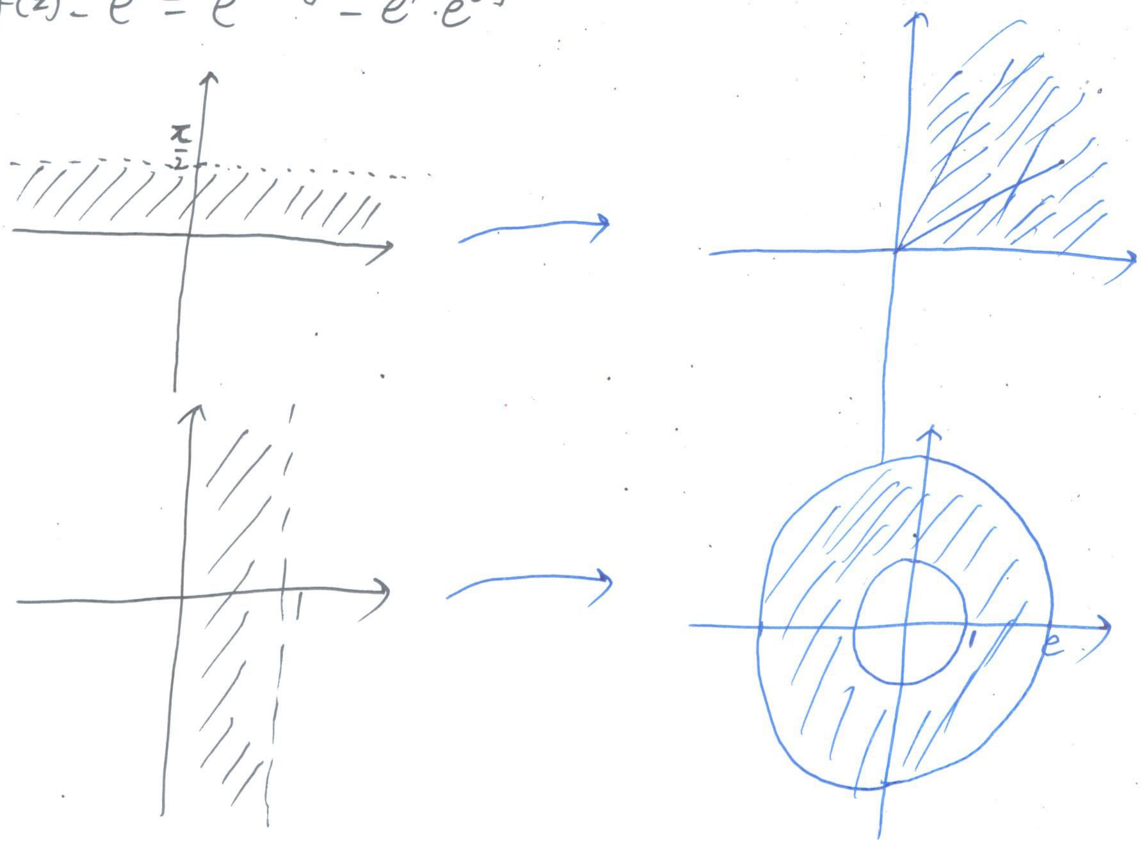
MATH2230 Tutorial 2

Mappings:

1. e^z:

$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$



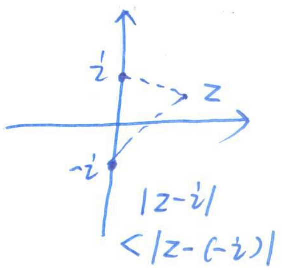
2. $\frac{z-i}{z+i}$: $H = \{x+iy : y>0\} \longrightarrow D = \{z : |z|<1\}$

1°. $\forall z \in H$, z is in the upper half plane

$\therefore |z-i| < |z+i|$

$\therefore \left|\frac{z-i}{z+i}\right| < 1$

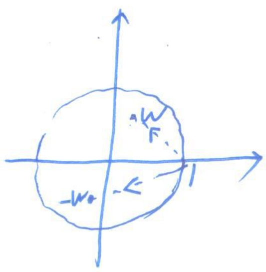
$\therefore f(z) = \frac{z-i}{z+i} \in D$ for $\forall z \in H$



$\frac{z-i}{z+i}$ is 1-1 and onto from H to D

2°. $\forall w \in D$, if $\frac{z-i}{z+i} = w$

then $z = \frac{1+w}{1-w} i = \frac{1-(-w)}{1-w} i$



$\therefore$ [redacted]

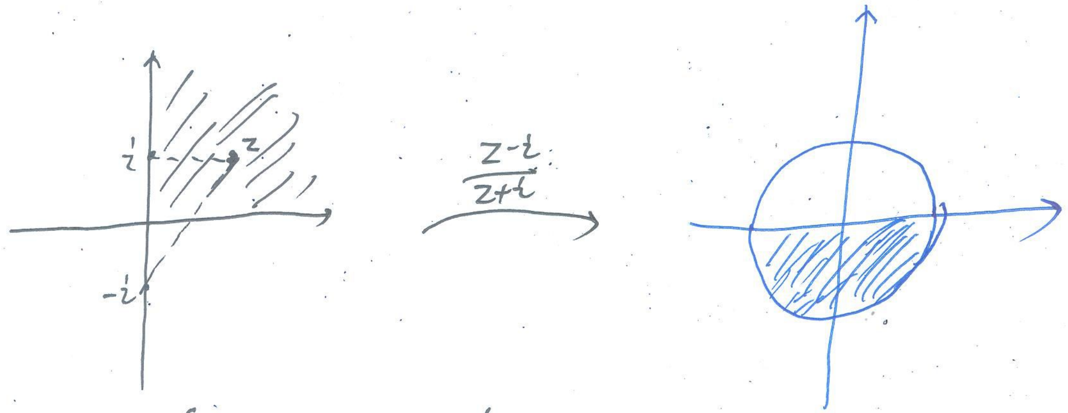
$\arg(z) = \arg\left(\frac{1-(-w)}{1-w} \cdot i\right)$

$= \arg(1-(-w)) - \arg(1-w) + \arg(i)$

$\in (-\frac{\pi}{2}, \frac{\pi}{2}) + \frac{\pi}{2} = (0, \pi)$

$\therefore z \in H$

3°、



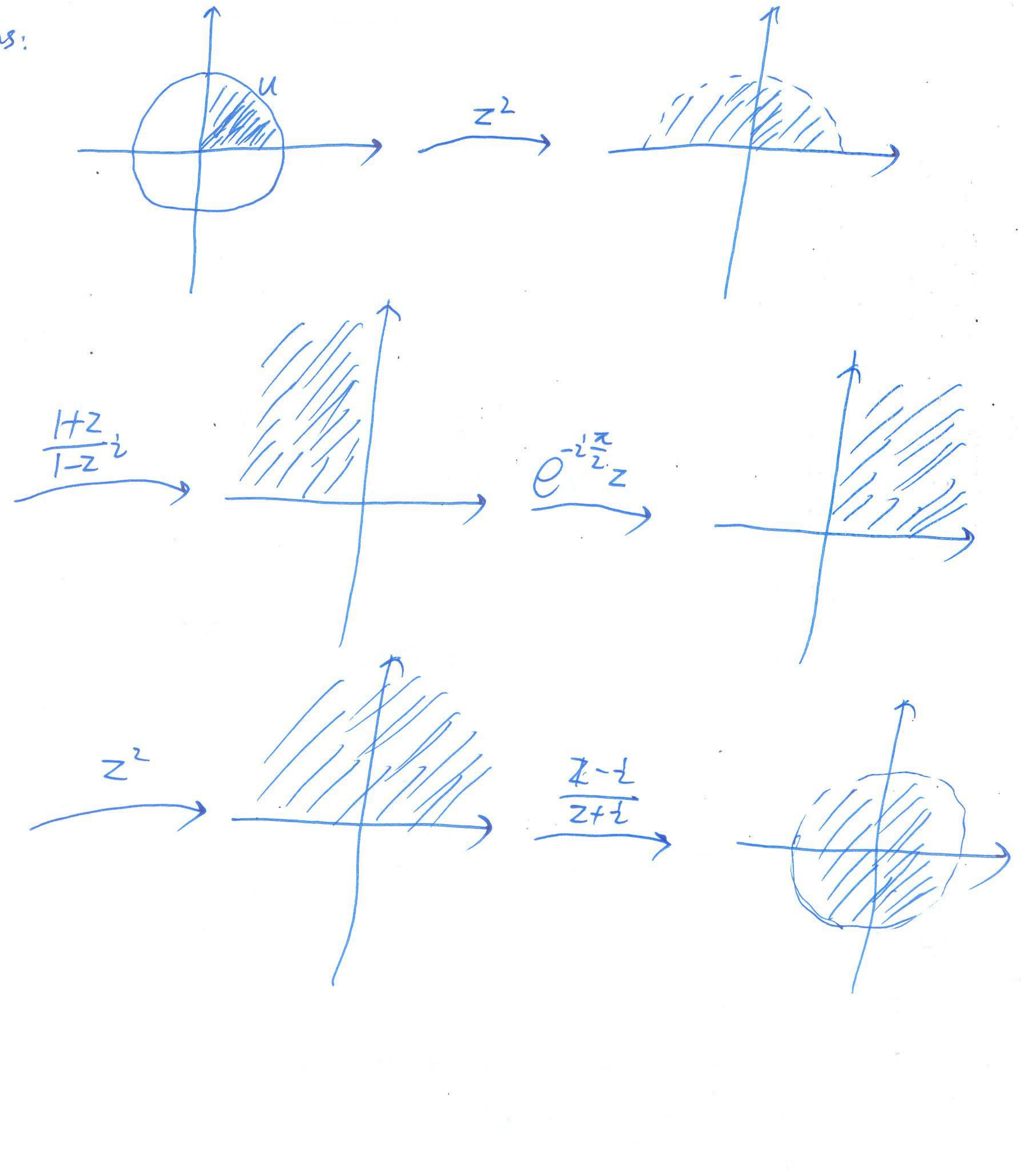
for $z \in \{x+iy:\ x>0, y>0\}$

$\arg\left(\frac{z-i}{z+i}\right) \in (-\pi, 0)$

3、Let $U = \{z \mid |z|<1,\ 0<\arg z<\frac{\pi}{2}\}$

Find a 1-1 and onto differentiable map from U to D.

Ans:



Cauchy-Riemann equations for differentiable complex functions.

- Def: $f(z)$ is differentiable at z_0 if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists

- So if we approach z_0 from z_0+t, z_0+ti directions the limits must be equal.

$$\lim_{t \to 0} \frac{f(z_0+t)-f(z_0)}{t} = \lim_{t \to 0} \frac{f(z_0+ti)-f(z_0)}{ti}$$

- Set $f(x+iy) = u(x,y) + i v(x,y)$

then
$$\lim_{t \to 0} \frac{[u(x_0+t, y_0)-u(x_0,y_0)] + i[v(x_0+t, y_0) - v(x_0, y_0)]}{t}$$
$$= \lim_{t \to 0} \frac{[u(x_0, y_0+t)-u(x_0,y_0)] + i[v(x_0, y_0+t) - v(x_0, y_0)]}{ti}$$

$$\therefore u_x + i v_x = \frac{u_y + i v_y}{i} = \frac{i u_y - v_y}{i^2} = v_y - i u_y$$

$$\therefore \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad (\text{C-R eqn})$$

- ~~$f'(z_0) = u_x$~~ $f'(z) = u_x + i v_x = v_y - i u_y$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{f} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

$$f' = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} = \sqrt{u_x^2+u_y^2} \begin{bmatrix} \frac{u_x}{\sqrt{u_x^2+u_y^2}} & \frac{u_y}{\sqrt{u_x^2+u_y^2}} \\ -\frac{u_y}{\sqrt{u_x^2+u_y^2}} & \frac{u_x}{\sqrt{u_x^2+u_y^2}} \end{bmatrix}$$

$$= \sqrt{u_x^2+u_y^2} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation}}$$

- Let $w = f(z)$

then locally, $dw \doteq f'(z)\,dz$

$\therefore$ A differentiable function is locally a rotation.

- show $f(z) = \bar{z}$ is not differentiable

4. Let c be a constant
let $f(z)$ be a differentiable function and $Re(f(z)) \equiv c$ i.e.
(If we set $f(z)=u+iv$, then $u \equiv c$)
show that f is a constant function.

pf: $\because$ $u(x,y) \equiv c$, c is a constant

$\therefore$ $u_x = u_y = 0$

$\because$ $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\therefore$ $v_x = v_y = 0$

$\therefore$ v is a constant

$\therefore$ $f = u+iv$ is a constant #.

• It means if f is differentiable, then there are some connections between its real part and imaginary part.