MATH 2230 Tutorial 12

1. How many zeros does the function $f(z) = 2z^2 - e^{\frac{z}{2}}$ have in $|z|<1$?

Ans: Let $g(z) = 2z^2$

$\therefore |f-g| = |e^{\frac{z}{2}}| = |e^{\frac{x+iy}{2}}| = e^{\frac{x}{2}} \leq e^{\frac{1}{2}}$ for $|z|=1$

< 2

$= |g|$ for $|z|=1$

$\therefore$ By Rouché's Thm

f and g have the same number of zeros inside $|z|<1$

$\because 2z^2$ has two zeros inside $|z|<1$

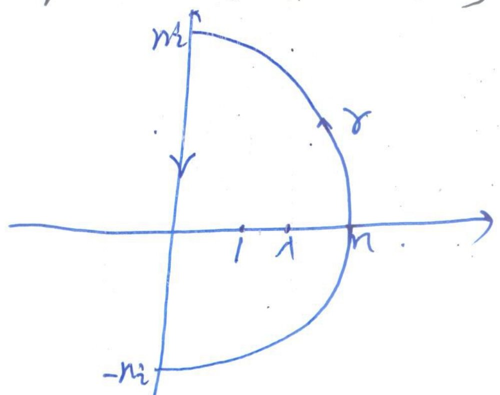
$\therefore f$ has two zeros inside $|z|<1$

2. Let $\lambda > 1$. Show that the equation

$$\lambda - z - e^{-z} = 0$$

has exactly one solution in the right half plane $\{z \mid Re(z) > 0\}$.

Also show that the solution is real.



Ans: Let $f(z) = \lambda - z - e^{-z}$

$g(z) = \lambda - z$

Let γ be the contour in the graph

$\therefore$ when n is large enough

$|g| = |\lambda - z| \geq \lambda > 1$ for $z \in \gamma$

$\therefore |f-g| = |e^{-z}| = |e^{-x-iy}| = e^{-x} \leq 1$ for $z \in \gamma$ since $Re(z) \geq 0$

$\therefore |f-g| \leq 1 < \lambda \leq |g|$ for $z \in \gamma$

$\therefore$ By Rouché's Thm

f and g have the same number of root inside γ

$\because g = \lambda - z$ has only one root inside γ

$\therefore f$ has only one root inside γ for n is large

$\therefore f$ has only one root on the right half plane

$\because f(0) = \lambda - 1 > 0$, $f(x) \to -\infty$ as $x \to \infty$ $\therefore$ By MVT, $\exists x_0 \in (0, +\infty)$ s.t $f(x_0) = 0$

3. Determine the number of zeros of the polynomial

$z^7 - 4z^3 + z - 1$ inside $|z|=1$

Ans: Let $f(z) = z^7 - 4z^3 + z - 1$

$g(z) = -4z^3$

$\because |f-g| = |z^7 + z - 1| \leq 1+1+1 = 3 < 4 = |g|$ for $|z|=1$

$\therefore$ By Rouché's Thm

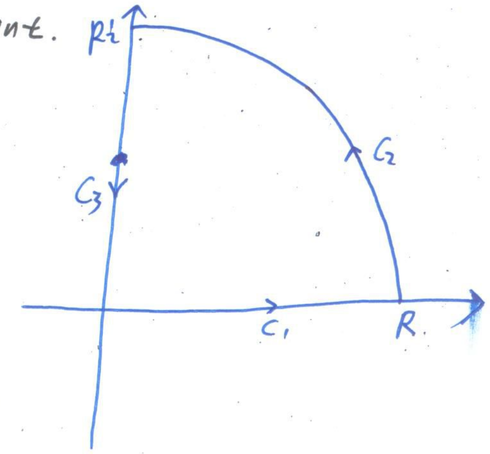
f and g have the same number of zeros inside $|z|=1$

$\because -4z^3$ has 3 zeros inside $|z|=1$

$\therefore f$ has 3 zeros inside $|z|=1$

4. Let $f(z) = z^4 + z^3 + 4z^2 + 2z + 3$

Show f has no zero in the first quadrant.



Ans: let $C_1 = \{x : x \in [0, R]\}$

$C_2 = \{Re^{i\theta} : \theta \in [0, \frac{\pi}{2}]\}$

$C_3 = \{yi : y \text{ from } R \text{ to } 0\}$

1°. for $z = Re^{i\theta}$

$$|f(z)| = |R^4 e^{i4\theta} + R^3 e^{i3\theta} + 4z^2 + 2z + 3|$$
$$\geq R^4 - R^3 - 4R^2 + 2R + 3 > 0 \quad \text{if } R \text{ is sufficient large.}$$

$\therefore$ we choose R large such that $f(z) \neq 0$ for $z \in C_2$

2°. for $z \in C_1$, $z = x$

$$f(z) = x^4 + x^3 + 4x^2 + 2x + 3 \geq 3$$

3°. for $z \in C_3$, $z = yi$

$$f(z) = (yi)^4 + (yi)^3 + 4(yi)^2 + 2(yi) + 3$$
$$= y^4 - y^3 i - 4y^2 + 2yi + 3$$
$$= (y^4 - 4y^2 + 3) + iy(-y^2 + 2)$$
$$= (y^2 - 1)(y^2 - 3) + iy(\sqrt{2} - y)(\sqrt{2} + y)$$
$$= (y-1)(y+1)(y-\sqrt{3})(y+\sqrt{3}) + iy(\sqrt{2} - y)(\sqrt{2} + y)$$

4°. • for $z \in C_3$

when $y \in (\sqrt{3}, R)$

$Re(f(z)) > 0$, $Im(f(z)) < 0$

and $\lim_{R \to \infty} \frac{Re(f(z))}{Im(f(z))} = \infty$

when $y \in (\sqrt{2}, \sqrt{3})$

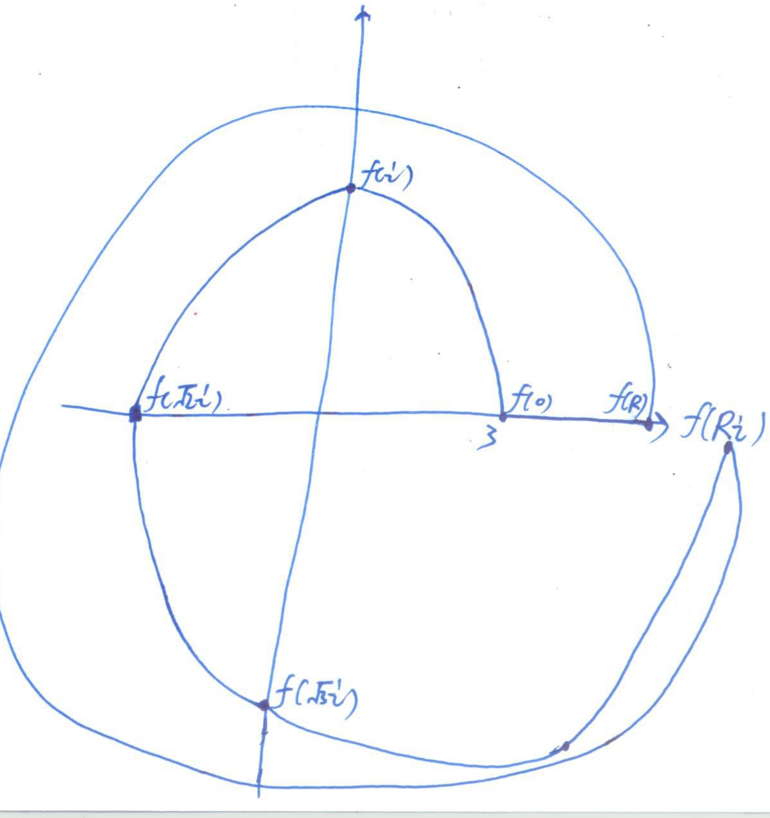
$Re(f(z)) < 0$, $Im(f(z)) < 0$

when $y \in (1, \sqrt{2})$

$Re(f(z)) < 0$, $Im(f(z)) > 0$

when $y \in (0, 1)$

$Re(f(z)) > 0$, $Im(f(z)) > 0$



• when $z \in C_2$

~~Arg~~ $Arg(f(z))$ increases 2π

$\therefore$ the winding number is 0.