MATH 2230 Tutorial 11

1. Find the residue at $z=0$ of $f(z)=\frac{e^{2z}}{z^4}$

Ans: Let $\phi(z)=e^{2z}$

$\therefore f(z)=\frac{\phi(z)}{z^4}$ and $\phi(0)\neq 0$

$\therefore \operatorname{Res}(f,0)$

$=\frac{\phi^3(0)}{3!}=\frac{2^3\cdot 1}{3!}=\frac{4}{3}$

2. Find the integral:

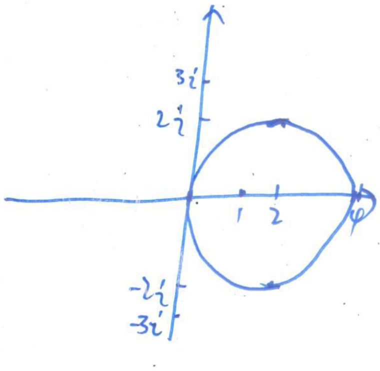
$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

taken counterclockwise around the circle

(a) $|z-2|=2$

(b) $|z|=4$

Ans: Let $f(z)=\frac{3z^3+2}{(z-1)(z^2+9)}$



(a) the poles of $f(z)$ inside $|z-2|=2$ is $z=1$

$\therefore \operatorname{Res}(f,1)=\frac{3+2}{1+9}=\frac{1}{2}$

$\therefore \int_{C_1} f(z)dz=2\pi i\cdot\frac{1}{2}=\pi i$

(b) the poles of $f(z)$ inside $|z|=4$ are

$z_1=1,\quad z_2=3i,\quad z_3=-3i$

$\therefore \operatorname{Res}(f,1)=\frac{3+2}{1+9}=\frac{1}{2}$

$\operatorname{Res}(f,3i)=\frac{3(3i)^3+2}{(3i-1)(3i+3i)}=\frac{-81i+2}{6i(3i-1)}$

$\operatorname{Res}(f,-3i)=\frac{3(-3i)^3+2}{(-3i-1)(-3i-3i)}=\frac{81i+2}{6i(3i+1)}$

$\therefore \sum_{i=1}^{3}\operatorname{Res}(f,z_i)=\frac{1}{2}+\frac{-81i+2}{6i(3i-1)}+\frac{81i+2}{6i(3i+1)}=\frac{1}{2}+\frac{5}{2}=3$

$\therefore \int_{C_2} f(z)dz=2\pi i\cdot 3=6\pi i$

· Remark: if z_0 is a pole of order 1 of $f(z)$

then $\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0)f(z)$

Pf: $f(z) = \frac{\phi(z)}{z-z_0}$ where $\phi(z_0) \neq 0$

$\therefore \operatorname{Res}(f, z_0) = \phi(z_0)$

$\because \lim_{z\to z_0} f(z)(z-z_0) = \lim_{z\to z_0} \frac{\phi(z)}{(z-z_0)}(z-z_0) = \phi(z_0)$

$\therefore \operatorname{Res}(f, z_0) = \phi(z_0) = \lim_{z\to z_0} (z-z_0)f(z)$

3、 Compute $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

using the Residue Theorem and the function

$$f(z) = \frac{\pi \csc(\pi z)}{z^2}$$

Ans: 1°、 the poles of $f(z)$ are $0, \pm 1, \pm 2, \pm 3, \cdots$

and $\pm 1, \pm 2, \pm 3 \cdots$ are simple poles of f

$\therefore \operatorname{Res}(f, n) = \lim_{z\to n} f(z)(z-n) = \lim_{z\to n} \frac{\pi(z-n)}{z^2 \sin \pi z} = \frac{\pi}{n^2} \cdot \lim_{z\to n} \frac{1}{\pi \cos \pi z} = \frac{(-1)^n}{n^2}$

for $n \neq 0$

$\because 0$ is a pole of order 3 of $f(z)$

~~$\therefore \operatorname{Res}(f, 0) = \lim_{z\to 0} \frac{(f(z)\cdot z^3)''}{2!}$~~

$\because \frac{1}{\sin \pi z} = \frac{1}{\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots} = \frac{1}{\pi z} \cdot \frac{1}{1 - (\frac{\pi^2 z^2}{3!} - \frac{\pi^4 z^4}{5!} + \cdots)}$

$= \frac{1}{\pi z}\left[1 + (\frac{\pi^2 z^2}{3!} - \frac{\pi^4 z^4}{5!} + \cdots) + (\frac{\pi^2 z^2}{3!} - \frac{\pi^4 z^4}{5!} + \cdots)^2\right]$

$\therefore f(z) = \frac{1}{z^3}\left[1 + \frac{\pi^2}{6} z^2 + O(|z|^4)\right] = \frac{1}{z^3} + \frac{\pi^2}{6z} + \cdots$

$\therefore \operatorname{Res}(f, 0) = \frac{\pi^2}{6}$

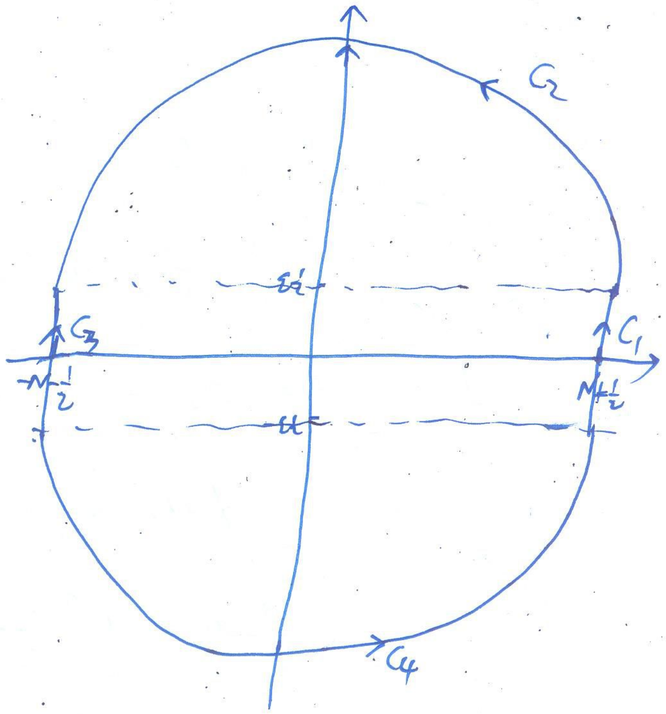
2, for $N \in \mathbb{N}$, $\frac{1}{2} > \varepsilon > 0$

let $C_1 = \{ N+\frac{1}{2} + yi : y \in [-\varepsilon, \varepsilon] \}$

$C_2 = \{ \varepsilon i + (N+\frac{1}{2}) e^{i\theta} : \theta \in [0, \pi] \}$

$C_3 = \{ -(N+\frac{1}{2}) + yi : y \in [-\varepsilon, \varepsilon] \}$

$C_4 = \{ -\varepsilon i + (N+\frac{1}{2}) e^{i\theta} : \theta \in [\pi, 2\pi] \}$



$\therefore$ By Residue Thm,

$$\int_{C_1+C_2-C_3+C_4} f(z)\,dz = 2\pi i \left(\frac{\pi^2}{6} + 2\sum_{i=1}^{N} \frac{(-1)^i}{i^2} \right)$$

• For $z \in C_1$, $z = N+\frac{1}{2} + yi$.

$$\therefore |f(z)| = \left| \frac{\pi}{z^2 \sin \pi z} \right| = \left| \frac{\pi}{(N+\frac{1}{2}+yi)^2} \cdot \frac{2i}{e^{i\pi(N+\frac{1}{2}+yi)} - e^{-i\pi(N+\frac{1}{2}+yi)}} \right|$$

$$\leq \frac{2\pi}{(N+\frac{1}{2}-\varepsilon)^2} \cdot \frac{1}{\left| e^{-y\pi}(-1)^N \cdot i - e^{y\pi}(-1)^{-N} \cdot (-i) \right|}$$

$$\leq \frac{2\pi}{N^2} \cdot \frac{1}{e^{y\pi} + e^{-y\pi}} \leq \frac{\pi}{N^2} \quad \text{since } e^{y\pi} + e^{-y\pi} \geq 2$$

similarly: for $z \in C_3$,

$$|f(z)| \leq \frac{4\pi}{N^2}$$

• For $z \in C_2$, $z = \varepsilon i + (N+\frac{1}{2}) e^{i\theta}$

$$|f(z)| = \left| \frac{\pi}{z^2 \sin \pi z} \right| = \left| \frac{\pi}{(\varepsilon i + (N+\frac{1}{2})e^{i\theta})^2} \cdot \frac{2i}{e^{i\pi[\varepsilon i + (N+\frac{1}{2})e^{i\theta}]} - e^{-i\pi[\varepsilon i + (N+\frac{1}{2})e^{i\theta}]}} \right|$$

$$\leq \frac{2\pi}{(N+\frac{1}{2}-\varepsilon)^2} \cdot \left| \frac{1}{e^{-\varepsilon\pi} e^{-\pi \sin\theta} e^{i[\pi(N+\frac{1}{2})\cos\theta]} - e^{\varepsilon\pi} e^{\pi \sin\theta} e^{-i[\cdots]}} \right|$$

$$\leq \frac{2\pi}{N^2} \cdot \frac{1}{e^{\varepsilon\pi} e^{\pi \sin\theta} - e^{\varepsilon\pi} e^{-\pi \sin\theta}}$$

$$\leq \frac{2\pi}{N^2} \cdot \frac{1}{e^{\varepsilon\pi} - e^{-\varepsilon\pi}} \quad \text{since } \sin\theta \geq 0 \text{ for } \theta \in [0, \pi]$$

$$\leq \frac{2\pi C_0}{N^2}$$

Similarly, for $z \in C_4$, $|f(z)| \leq \frac{2\pi C_0}{N^2}$

3°. $\left|\int_{C_1+C_2-C_3+C_4} f(z)\,dz\right|$

$$\le \left|\int_{C_1} f(z)\,dz\right| + \left|\int_{C_2} f(z)\,dz\right| + \left|\int_{C_3} f(z)\,dz\right| + \left|\int_{C_4} f(z)\,dz\right|$$

$$= \left|\int_{-\xi}^{\xi} f(z)\, i\,dy\right| + \left|\int_{0}^{\pi} f(z)\, i\left(N+\tfrac{1}{2}\right) e^{i\theta}\,d\theta\right|$$

$$+ \left|\int_{-\xi}^{\xi} f(z)\, i\,dy\right| + \left|\int_{\pi}^{2\pi} f(z)\, i\left(N+\tfrac{1}{2}\right) e^{i\theta}\,d\theta\right|$$

$$\le \int_{-\xi}^{\xi} \frac{4\pi}{N^2}\,dy + \int_{0}^{\pi} \frac{2\pi C_0}{N^2}\cdot\left(N+\tfrac{1}{2}\right)d\theta$$

$$+ \int_{-\xi}^{\xi} \frac{4\pi}{N^2}\,dy + \int_{\pi}^{2\pi} \frac{2\pi C_0}{N^2}\left(N+\tfrac{1}{2}\right)d\theta$$

$$= \frac{16\pi\xi}{N^2} + \frac{4\pi^2 C_0}{N^2}\left(N+\tfrac{1}{2}\right) \to 0 \quad \text{as } N\to\infty$$

$$\therefore \lim_{N\to\infty}\left(\frac{\pi^2}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2}\right) = 0$$

$$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$