MATH 2230 Tutorial 10

1. Let f be analytic in $|z|<1$.
   Suppose $|f(\frac{1}{n})| \leq \alpha^{-n}$ for some $\alpha > 1$ for all $n = 2, 3, 4, \cdots$
   Show $f \equiv 0$

Ans: $\because f$ is continuous

$\therefore |f(0)| = \lim_{n\to\infty} |f(\frac{1}{n})| \leq \lim_{n\to\infty} \frac{1}{\alpha^n} = 0$

$\therefore f(0) = 0$

$\because f$ is analytic in $|z| \leq 1$

~~$\therefore f = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ in $|z|<1$~~ [Assume f is not identically 0]

$\exists\ k_0 \geq 1$ such that

$f(z) = z^{k_0} g(z)$ where $g(z)$ is analytic on $|z|<1$ and $g(0) \neq 0$

$\because |f(\frac{1}{n})| \leq \frac{1}{\alpha^n}$

$\therefore \frac{1}{n^{k_0}} |g(\frac{1}{n})| \leq \frac{1}{\alpha^n}$

$\therefore \alpha^n |g(\frac{1}{n})| \leq n^{k_0}$

$\because g(0) \neq 0$, g is continuous

$\therefore \exists N_0 \in \mathbb{N}$ such that

$\frac{|g(0)|}{2} < |g(\frac{1}{n})| < 2|g(0)|$ for $\forall n > N_0$

$\therefore \ln(\alpha^n |g(\frac{1}{n})|) \leq \ln(n^{k_0})$

$n \ln\alpha + C \leq n \ln\alpha + \ln|g(\frac{1}{n})| \leq k_0 \ln n$

$\therefore n \leq \frac{k_0 \ln n - C}{\ln\alpha}$ where $k_0 > 1$, $\alpha > 1$ for $\forall n > N_0$

it is a contradiction since n increases much faster than $\ln n$

$\therefore f$ is identically 0.

2. Let f be an entire function

Let $g(z) = \frac{a}{z} + f(z)$ where a is a constant

Suppose $g(z) \in R$ for $\forall\ |z|=1$

Show that $g(z) = \frac{a}{z} + b + \bar{a}z$ for some $b \in R$

Ans: $\because f(z)$ is entire

$\therefore f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ for $\forall\ z \in \mathbb{C}$

Let $h(z) = \overline{g(\frac{1}{\bar{z}})}$

$$\therefore h(z) = \overline{\left(\frac{a}{\frac{1}{\bar{z}}} + a_0 + a_1 \cdot \frac{1}{\bar{z}} + a_2 \cdot \frac{1}{\bar{z}^2} + \cdots\right)}$$

$$= \bar{a}z + \bar{a}_0 + \bar{a}_1 \cdot \frac{1}{z} + \bar{a}_2 \cdot \frac{1}{z^2} + \bar{a}_3 \cdot \frac{1}{z^3} + \cdots$$

$\therefore h(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ by the convergence of the power series

- For $|z|=1$, $\frac{1}{\bar{z}} = \frac{1}{e^{-i\theta}} = e^{i\theta} = z$

$\because g(z) \in R$

$\therefore h(z) = \overline{g(\frac{1}{\bar{z}})} = \overline{g(z)} = g(z)$

$\therefore h(z) = g(z)$ on $|z|=1$

$\therefore$ By result of Tutorial 9

$$h(z) \equiv g(z) \Rightarrow \bar{a}z + \bar{a}_0 + \bar{a}_1 \cdot \frac{1}{z} + \bar{a}_2 \cdot \frac{1}{z^2} + \cdots = \frac{a}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$$

Then by the uniqueness of Laurent series

$$\begin{cases} a_2 = a_3 = a_4 = \cdots = 0 \\ \bar{a}_0 = a_0 \\ \bar{a}_1 = \bar{a} \end{cases}$$

$\therefore g(z) = \frac{a}{z} + a_0 + \bar{a}z$

3. Find the Laurent series of the function $f(z)=\frac{1}{4z-z^2}$ in the region

i) $\{z\in\mathbb{C} \mid 0<|z|<4\}$

ii) $\{z\in\mathbb{C} \mid |z|>4\}$

Ans: i) when $0<|z|<4$

$$f(z)=\frac{1}{4z-z^2}=\frac{1}{4z}\cdot\frac{1}{1-\frac{z}{4}}=\frac{1}{4z}\left(1+\frac{z}{4}+\left(\frac{z}{4}\right)^2+\cdots\right)$$

$$=\frac{1}{4z}+\frac{1}{4^2}+\frac{z}{4^3}+\frac{z^2}{4^4}+\cdots$$

$$=\frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$$

ii) when $|z|>4$

$$f(z)=\frac{1}{4z-z^2}=-\frac{1}{z^2}\cdot\frac{1}{1-\frac{4}{z}}=-\frac{1}{z^2}\left(1+\frac{4}{z}+\frac{4^2}{z^2}+\frac{4^3}{z^3}+\cdots\right)$$

$$=-\sum_{n=0}^{\infty}\frac{4^n}{z^{n+2}}$$

4. Let $f(z) = \frac{e^z}{z(z^2+1)}$. Find the following terms in the Laurent series expansion

i) first three positive degree terms in $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

ii) first negative degree term in $\{z \in \mathbb{C} \mid |z| > 1\}$

Ans: i) when $0 < |z| < 1$

$$f(z) = \frac{e^z}{z(z^2+1)} = \frac{1}{z} \cdot \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 - z^2 + z^4 - z^6 + \cdots\right)$$

$\therefore$ degree 1 term $= \frac{1}{z}\left(-z^2 + \frac{z^2}{2!}\right) = -\frac{z}{2}$.

degree 2 term $= \frac{1}{z}\left(\frac{z^3}{3!} - z^3\right) = -\frac{5}{6}z^2$

degree 3 term $= \frac{1}{z}\left(\frac{z^4}{4!} - \frac{z^4}{2!} + z^4\right) = \frac{13}{24}z^3$

ii) when $|z| > 1$

$$f(z) = \frac{e^z}{z(z^2+1)} = \frac{e^z}{z^3} \cdot \frac{1}{1 + \frac{1}{z^2}} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right)$$

$\therefore$ degree (-1) term

$$= \frac{1}{z^3}\left(\frac{z^2}{2!} \cdot 1 + \frac{z^4}{4!}\left(-\frac{1}{z^2}\right) + \frac{z^6}{6!}\left(\frac{1}{z^4}\right) + \cdots\right)$$

$$= \frac{1}{z^3} \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)!} \cdot (-1)^{n-1} \frac{1}{z^{2n-2}}$$

$$= \frac{1}{z^3} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n)!} \cdot z^2$$

$$= \frac{1}{z}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}\right)$$