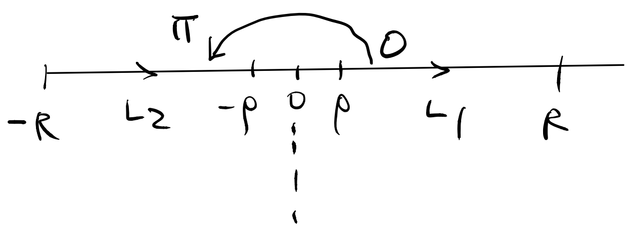
For (4), we cannot simply put x for z as we need to take care the branch of $\log z$

On L_1:

$z = re^{i\theta}$ with



$\rho \le r \le R$ and $\theta \equiv 0$

L_1 can be parametrized by $z = r$, $\rho \le r \le R$ and

$$\log z = \ln r \quad \text{on } L_1 \quad (\text{because } \theta \equiv 0 \text{ for this branch})$$

$$\therefore \int_{L_1} f(z)\,dz = \int_\rho^R \frac{e^{a \ln r}}{(r^2+1)^2}\,dr = \int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr$$

However on L_2,

$$z = re^{i\theta} \text{ with } \rho \le r \le R \text{ and } \theta \equiv \pi$$

$\therefore$ $-L_2$ can be parametrized by $z = re^{i\pi} = -r$, $\rho \le r \le R$

and $\log z = \ln r + i\pi$, on L_2

$$\therefore \int_{L_2} f(z)\,dz = -\int_{-L_2} \frac{e^{a \log z}}{(z^2+1)^2}\,dz = -\int_\rho^R \frac{e^{a(\ln r + i\pi)}}{(r^2+1)^2}\,(-dr)$$

$$= \int_{\rho}^{R} \frac{r^a e^{ia\pi}\, dr}{(r^2+1)^2} = e^{ia\pi} \int_{\rho}^{R} \frac{r^a\, dr}{(r^2+1)^2}$$

Hence $\displaystyle\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = (1+e^{ia\pi}) \int_{\rho}^{R} \frac{r^a dr}{(r^2+1)^2}$

$$\Rightarrow \lim_{\substack{R\to\infty \\ \rho\to 0}} \left[\int_{L_1} f(z)dz + \int_{L_2} f(z)dz \right] = (1+e^{ia\pi}) \int_0^{\infty} \frac{r^a dr}{(r^2+1)^2}$$

All together, we have

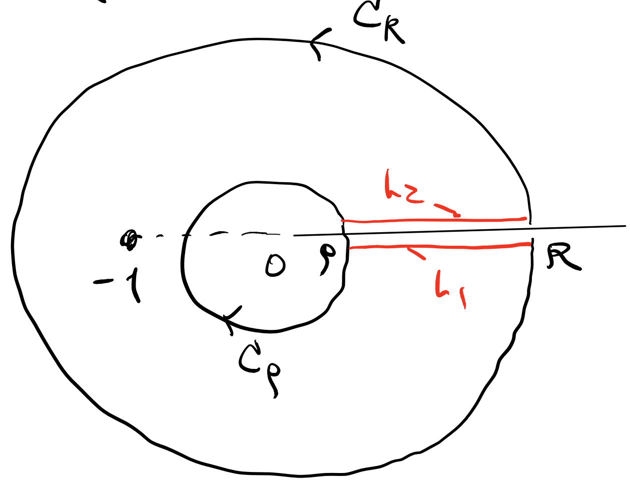
$$(1+e^{ia\pi}) \int_0^{\infty} \frac{r^a dr}{(r^2+1)^2} = 2\pi i \left(-i e^{i\frac{a\pi}{2}} \cdot \frac{1-a}{4} \right)$$

$$\Rightarrow \int_0^{\infty} \frac{r^a dr}{(r^2+1)^2} = \frac{2\pi e^{i\frac{a\pi}{2}}}{(1+e^{ia\pi})} \frac{(1-a)}{4}$$

$$= \frac{(1-a)\pi}{4\cos(\frac{a\pi}{2})} \quad . \#$$

§91 Integration Along a Branch Cut

eg: Evaluate $\displaystyle\int_0^\infty \frac{x^{-a}}{1+x}\,dx$ for $0<a<1$



Soln: Consider

$$f(z) = \frac{z^{-a}}{1+z}$$

with the branch of $z^{-a} = e^{-a\log z}$

with $\log z = \ln r + i\theta$, $0<\theta<2\pi$

Consider the contour

$$\Gamma = C_R + L_1 + C_\rho + L_2$$ (in limiting sense as L_1, L_2 approaching the horizontal axis.)

Then on L_1:

$$\log z = \ln r + 2\pi i \quad (\rho \le r \le R)$$

and on L_2:

$$\log z = \ln r + 0\cdot i \quad (\rho \le r \le R)$$
$$= \ln r$$

$$\Rightarrow \int_{L_1} f(z)\,dz = -\int_\rho^R \frac{e^{-a(\ln r + 2\pi i)}}{1+r}\,dr \qquad (L_1 \text{ in negative direction})$$

$$= -e^{-2a\pi i}\int_\rho^R \frac{r^{-a}}{1+r}\,dr$$

$$\& \int_{L_2} f(z)\,dz = \int_\rho^R \frac{e^{-a\ln r}}{1+r}\,dr = \int_\rho^R \frac{r^{-a}}{1+r}\,dr$$

$$\therefore \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \left(1 - e^{-2a\pi i}\right)\int_\rho^R \frac{r^{-a}}{1+r}\,dr$$

Also $$\left|\int_{C_\rho} f(z)\,dz\right| = \left|\int_{C_\rho} \frac{z^{-a}}{1+z}\,dz\right| \le \frac{\rho^{-a}}{1-\rho}\cdot 2\pi\rho$$

$$= \frac{2\pi}{1-\rho}\,\rho^{1-a} \to 0 \quad \text{as } \rho \to 0$$

$$(0 < a < 1)$$

and $$\left|\int_{C_R} f(z)\,dz\right| = \left|\int_{C_R} \frac{z^{-a}}{1+z}\,dz\right| \le \frac{R^{-a}}{R-1}\cdot 2\pi R$$

$$= \frac{2\pi}{R^a}\cdot\frac{R}{R-1} \to 0 \quad \text{as } R \to \infty$$

$$(0 < a < 1)$$

Hence, Cauchy Residue Thm $\Rightarrow$

$$\left(1-e^{-2a\pi i}\right)\int_0^\infty \frac{x^{-a}}{1+x}\,dx = 2\pi i \operatorname*{Res}_{z=-1}\left(\frac{z^{-a}}{1+z}\right)$$

$$= 2\pi i\, e^{-a\pi i} \quad (\text{Ex!})$$

$$\therefore \int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{2\pi i\, e^{-a\pi i}}{1-e^{-2a\pi i}} = \frac{\pi}{\sin(a\pi)}$$

#

§92 Definite Integrals Involving Sines and Cosines

For integrals of the type $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$,

consider

$$\int_{|z|=1} F\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right) \frac{dz}{iz}.$$

$$\left(\text{since } \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta\right)$$

eg1: $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1<a<1)$

Soln: If $a=0$, $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_0^{2\pi} d\theta = 2\pi$

we are done.

If $a \neq 0$, take $z=e^{i\theta}$,

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_{|z|=1} \frac{1}{1+a\left(\frac{z-\frac{1}{z}}{2i}\right)} \cdot \frac{dz}{iz}$$

$$\overset{(Ex)}{=} \int_{|z|=1} \frac{2}{az^2+2iz-a}\,dz$$

Poles are $\quad z = \dfrac{-1 \pm \sqrt{1-a^2}}{a} i \quad \left(\text{since } -1 < a < 1 \atop a \neq 0\right)$

Check $\quad \left| \dfrac{-1-\sqrt{1-a^2}}{a} i \right| = \dfrac{1+\sqrt{1-a^2}}{|a|} > 1$

& $\quad \left| \dfrac{-1+\sqrt{1-a^2}}{a} i \right| = \dfrac{|a|}{1+\sqrt{1-a^2}} < 1$

$\therefore \quad z_0 = \dfrac{-1+\sqrt{1-a^2}}{a} i$ is the only pole of

$$f(z) = \frac{2}{a(z-z_0)\left(z + \frac{1+\sqrt{1-a^2}}{a} i\right)}$$

inside $|z|=1$ with residue

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{2}{a\left(\frac{-1+\sqrt{1-a^2}}{a} i + \frac{1+\sqrt{1-a^2}}{a} i \right)}$$

$$= \frac{1}{i\sqrt{1-a^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$

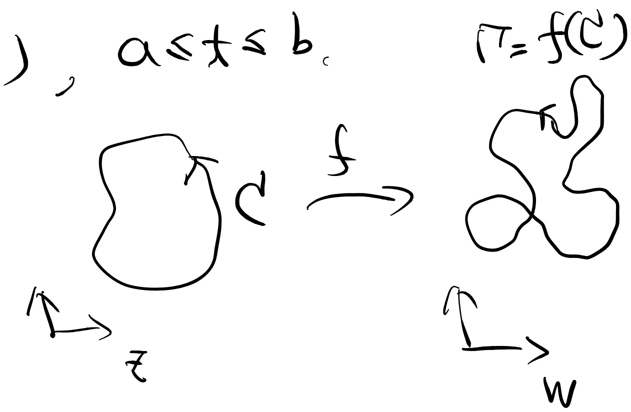
#

§ 93 Argument Principle

Def: A function is said to be meromorphic in a domain D if it is analytic throughout D except for poles.

Def: Let C be a positively oriented simple closed contour, and f a function meromorphic in the interior of C; analytic and nonzero on C. If C is parametrized by $z = z(t)$, $a \le t \le b$. Then the image of the contour C under f is a closed contour $\Gamma = f(C)$ parametrized by

$$w(t) = f(z(t)), \quad a \le t \le b.$$



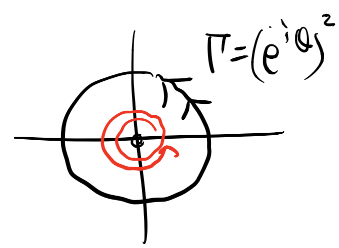
Express $w(t)$ as $f(z(t)) = w(t) = |w(t)|\, e^{i\phi(t)}$ where $\phi(t)$ is a continuous choice of the argument of $w(t)$ for $a \le t \le b$. Then

$$\frac{1}{2\pi} \Delta_C \arg f(z) \stackrel{\text{def}}{=} \frac{1}{2\pi}\left[\phi(b) - \phi(a)\right]$$

is an integer called the winding number of Γ with respect to the origin $w=0$.

eg: $C = \{ z(t) = e^{it},\ 0 \le t \le 2\pi \}$

$f(z) = z^2$



Then $\Gamma = f(C)$ is parametric by

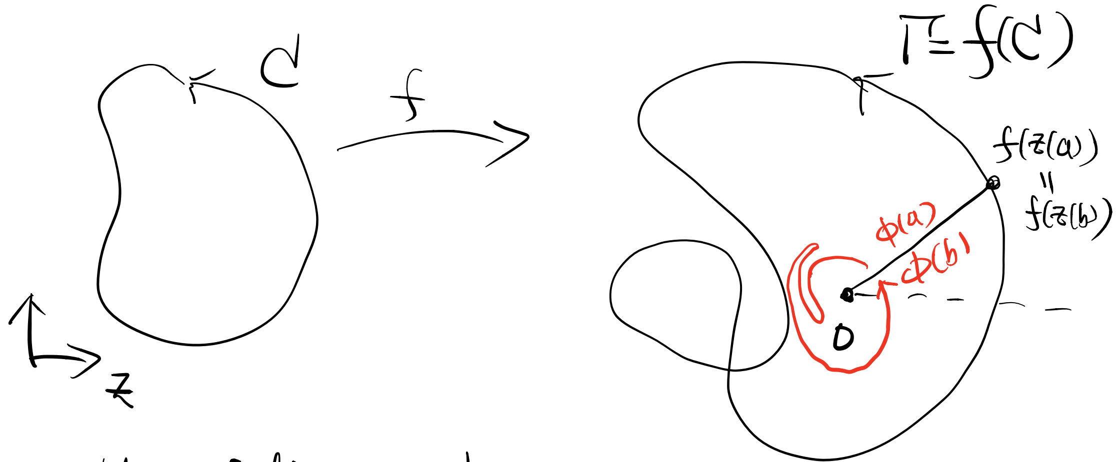
$$w(t) = f(z(t)) = (e^{it})^2 = e^{i2t}$$

Since $\phi(t) = 2t$, $0 \le t \le 2\pi$, is continuous, the winding number of Γ wrt $w=0$ is

$$\frac{1}{2\pi} \Delta_C \arg z^2 = \frac{1}{2\pi}\left[\phi(2\pi) - \phi(0)\right] = \frac{1}{2\pi}(4\pi - 0)$$

$$= 2.$$

<u>Note</u>: As $f(z) \neq 0$ for $z \in C$, $0 \notin \Gamma = f(C)$



Then the winding number of Γ can be interpreted as the <u>number of times</u> that <u>Γ surrounding</u> $w=0$.

<u>Thm</u>: Let C denote a positively oriented simple closed contour, and suppose that

(a) a function $f(z)$ is meromorphic in the domain interior to C;

(b) $f(z)$ is analytic and nonzero on C;

(c) counting multiplicities, Z = number of zeros and P = number of poles of f inside C.

Then $\boxed{\frac{1}{2\pi} \Delta_C \arg f(z) = Z - P}$.

Pf: Parametrize $\Gamma = f(C)$ as in the definition

$$w(t) = f(z(t)) = |w(t)|\, e^{i\phi(t)}, \quad a \le t \le b$$

Then
$$\int_C \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(z(t))\, z'(t)\,dt}{f(z(t))}$$

$$= \int_a^b \frac{\frac{d}{dt}\left(f(z(t))\right)}{|w(t)|\, e^{i\phi(t)}}\,dt$$

$$= \int_a^b \frac{\frac{d}{dt}|w(t)| \cdot e^{i\phi(t)} + i\,|w(t)|\, e^{i\phi(t)} \frac{d}{dt}\phi(t)}{|w(t)|\, e^{i\phi(t)}}\,dt$$

$$= \int_a^b \frac{\frac{d}{dt}|w(t)|}{|w(t)|}\,dt + i\int_a^b \frac{d}{dt}\phi(t)\,dt$$

$$= \left[\cancelto{0}{\ln|w(t)|}\right]_a^b + i\left[\phi(b) - \phi(a)\right]$$

$$= i\,\Delta_C \arg f(z).$$

$$\therefore\ \frac{1}{2\pi}\Delta_C \arg f(z) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

Note that $\frac{f'(z)}{f(z)}$ has isolated singular points at zeros and poles.

If z_k = zero of f of order m_k,

then $f(z) = (z - z_k)^{m_k} g(z)$

where g analytic & $\neq 0$ at z_k

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{m_k}{z - z_k} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{analytic at } z_k}$$

$$\Rightarrow \quad \operatorname*{Res}_{z=z_k} \frac{f'(z)}{f(z)} = m_k .$$

$$\therefore \quad \sum_k \operatorname*{Res}_{z=z_k} \frac{f'(z)}{f(z)} = \sum_k m_k = Z$$

number of zeros counting multiplicities.

Similarly for poles z_ℓ of order n_ℓ,

$$f(z) = \frac{g(z)}{(z - z_\ell)^{n_\ell}} , \quad g \text{ analytic } \& \neq 0 \text{ at } z_\ell$$

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{-n_\ell}{z - z_\ell} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{analytic at } z_\ell}$$

$$\Rightarrow \quad \operatorname*{Res}_{z=z_\ell} \frac{f'(z)}{f(z)} = -n_\ell$$

$\therefore \quad \sum_{\ell} \operatorname{Res}_{z=z_\ell} \frac{f'(z)}{f(z)} = \sum_{\ell} (-n_\ell) = -P$

the negative of the number of poles counting multiplicities.

Hence Cauchy Residue Thm $\Rightarrow$

$$\frac{1}{2\pi} \Delta_C \arg f(z) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

$$= \sum \text{Residues}$$

$$= Z - P \quad \divideontimes$$

eg: $f(z) = \frac{z^3+2}{z}$, $C = \{|z|=1\}$

Then f has only one simple pole at $z=0$ and no other pole & zero inside C

Hence, by argument principle,

$$\frac{1}{2\pi} \Delta_C \arg f(z) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -1.$$

$\Rightarrow$ the image contour $\Gamma = f(C)$ surrounds the origin once in negative direction. $\divideontimes$

§94 Rouché's Theorem

Thm (Rouché) Let C be a simple closed contour and suppose

(a) two functions $f(z)$ and $g(z)$ are analytic inside and on C;

(b) $|f(z)| > |g(z)|$ at each point $z \in C$.

Then $f(z)$ and $f(z)+g(z)$ have the <u>same</u> number of zeros, counting multiplicities, inside C.