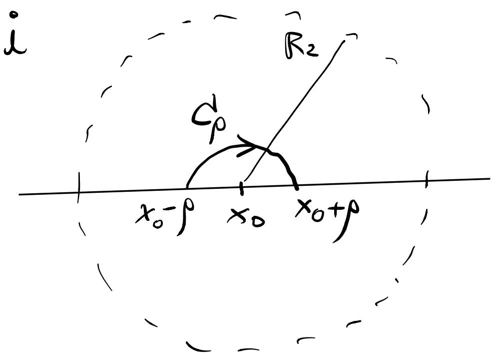
§89 An Indent Path

Thm Suppose that

(a) a function $f(z)$ has a simple pole at a point $z = x_0$ on the real axis, with a Laurent series representation in a punctured disk $0 < |z - x_0| < R_2$ and with residue B_0.

(b) C_ρ = negatively oriented upper semi-circle $|z - x_0| = \rho$ with $\rho < R_2$

Then $\displaystyle\lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = -B_0 \pi i$



Pf: By assumption

$$f(z) = \frac{B_0}{z - x_0} + a_0 + a_1(z - x_0) + \cdots$$

$$0 < |z - x_0| < R_2$$

$$\Rightarrow \int_{C_\rho} f(z)\,dz = B_0 \int_{C_\rho} \frac{dz}{z - z_0} + \int_{C_\rho} g(z)\,dz$$

where $\quad g(z) = a_0 + a_1(z-z_0) + \cdots$ is analytic in

$$|z - x_0| < R_2$$

By continuity of analytic function $g(z)$, there exists constant $M > 0$ and $\rho_0 < R_2$ such that

$\forall\ \rho < \rho_0$,

$$|g(z)| \le M \ , \quad \forall\ |z - x_0| \le \rho .$$

Hence $\quad \left| \int_{C_\rho} g(z)\,dz \right| \le M\pi\rho \to 0$ as $\rho \to 0$.

On the other hand, $\forall\ \rho < R_2$

$$\int_{C_\rho} \frac{dz}{z - x_0} = - \int_{-C_\rho} \frac{dz}{z - x_0} = - \int_0^\pi \frac{d(\rho e^{i\theta})}{\rho e^{i\theta}}$$

$$= - \int_0^\pi \frac{i \cancel{\rho e^{i\theta}}\, d\theta}{\cancel{\rho e^{i\theta}}}$$

$$= -\pi i$$

negatively oriented

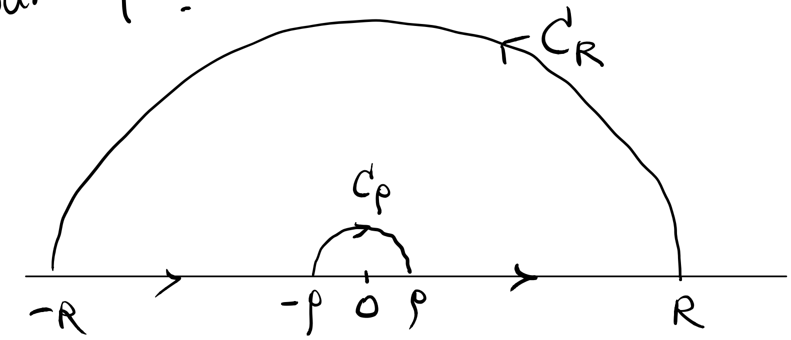
$$\therefore \quad \lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = -B_0 \pi i$$

⁂

eg (Dirichlet's Integral)

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Soln: Consider integral of $f(z) = \frac{e^{iz}}{z}$ on the contour Γ:



Cauchy integral formula $\Rightarrow$

$$0 = \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx + \int_{C_\rho} \frac{e^{iz}}{z}\,dz + \int_\rho^R \frac{e^{ix}}{x}\,dx$$

Since $\int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx = -\int_\rho^R \frac{e^{-ix}}{x}\,dx$ (Ex!)

$$\int_\rho^R \frac{e^{ix}}{x}\,dx + \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx = \int_\rho^R \frac{e^{ix}}{x}\,dx - \int_\rho^R \frac{e^{-ix}}{x}\,dx$$

$$= \int_\rho^R \frac{2i\sin x}{x}\,dx .$$

On the other hand, $\left|\frac{1}{z}\right| \le \frac{1}{R} = M_R \to 0$, $\forall z \in C_R$

Jordan's Lemma $\Rightarrow$ $\lim\limits_{R\to\infty} \int_{C_R} \frac{1}{z} e^{iz} dz = 0$.

Finally, $\frac{e^{iz}}{z} = \frac{1}{z}\left[1 + iz + \frac{(iz)^2}{2!} + \cdots\right]$

$$= \frac{1}{z} + i + \cdots \quad (0 < |z| < \infty)$$

has a simple pole at $z=0$ with residue

$B_0 = 1$.

$\therefore$ The Thm $\Rightarrow$ $\lim\limits_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}}{z} dz = -1 \cdot \pi i$

Hence by taking limit as $R\to\infty$ & $\rho \to 0$,

$$0 = 0 + \lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \frac{2i \sin x}{x} dx - \pi i$$

$$\therefore \quad \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

§90 An Indentation Around a Branch Point

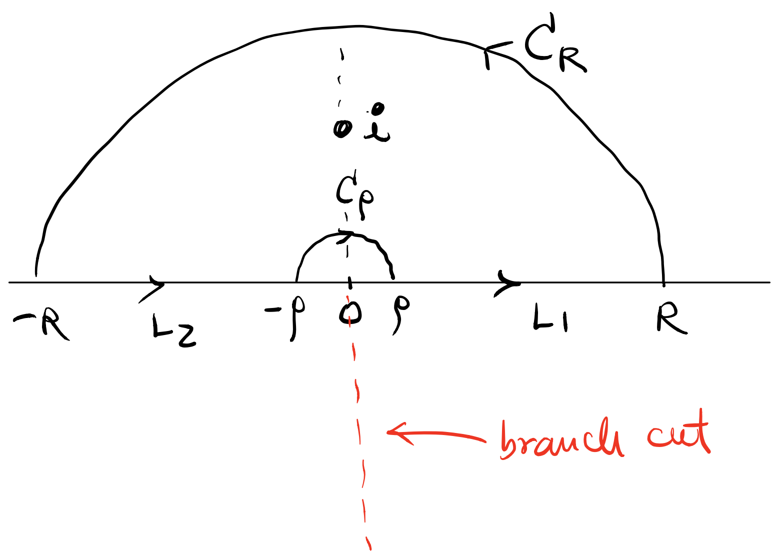
eg: $$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{(1-a)\pi}{4\cos\left(\frac{a\pi}{2}\right)} \qquad (-1<a<3)$$

Soln: Note $x^a = e^{a\ln x}$ for >0

$\therefore$ corresponding complex valued function should be

$$f(z) = \frac{e^{a\log z}}{(z^2+1)^2}$$ with suitable branch of $\log z$.

Γ:



For contour Γ as above, one cannot use the principal branch as it is not defined on $[-R,-\rho]$. Instead we can use the branch

$$\log z = \ln r + i\theta, \quad -\frac{\pi}{2}<\theta<\frac{3\pi}{2}.$$

With this branch $f(z) = \dfrac{e^{a \log z}}{(z^2+1)^2}$ only has an isolated singular point at $z_0 = i$ inside Γ.

Hence Cauchy Residue Thm $\Rightarrow$

$$\int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz + \int_{L_1} f(z)\,dz$$

$$= 2\pi i \operatorname*{Res}_{z_0 = i} f(z).$$

So we need to calculate

(1) $\operatorname*{Res}_{z_0=i} f(z)$,

(2) $\lim\limits_{R\to\infty} \int_{C_R} f(z)\,dz$,

(3) $\lim\limits_{\rho\to 0} \int_{C_\rho} f(z)\,dz$, and

(4) $\lim\limits_{\substack{R\to\infty \\ \rho\to 0}} \left[\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz \right]$

For (1), $f(z) = \dfrac{e^{a\log z}}{(z^2+1)^2} = \dfrac{1}{(z-i)^2} \cdot \dfrac{e^{a\log z}}{(z+i)^2}$

$$= \frac{\phi(z)}{(z-i)^2}$$

where $\phi(z) = \dfrac{e^{a \log z}}{(z+i)^2}$ analytic at i

and $\phi(i) = \dfrac{e^{a \log i}}{(i+i)^2}$

$$= \frac{e^{a(\ln 1 + i\frac{\pi}{2})}}{(2i)^2}$$

$$= \frac{e^{i\frac{a\pi}{2}}}{-4} \neq 0$$

$\therefore$ $z_0 = i$ is a pole of order 2

Hence $\underset{z_0=i}{\operatorname{Res}} f(z) = \dfrac{\phi^{(1)}(i)}{1!} = \dfrac{d}{dz}\Big|_{z=i} \dfrac{e^{a \log z}}{(z+i)^2}$

$$\overset{(Ex!)}{=} -i e^{\frac{ia\pi}{2}} \cdot \frac{1-a}{4}$$

For (2) $\left| \int_{C_R} f(z)\, dz \right| = \left| \int_{C_R} \dfrac{e^{a \log z}}{(z^2+1)^2} dz \right|$

$$\le \left| \int_{C_R} \frac{z^a}{(z^2+1)^2} dz \right|$$

$$\leq \frac{R^a}{(R^2-1)^2} \cdot \pi R$$

$$= \frac{\pi R^{1+a}}{R^4 \left(1-\frac{1}{R^2}\right)^2}$$

$$= \frac{\pi}{R^{3-a}\left(1-\frac{1}{R^2}\right)^2}$$

$\to 0$ as $R \to \infty$

$(-1 < a < 3)$

For (3) $\left| \int_{C_\rho} f(z)\, dz \right| = \left| \int_{C_\rho} \frac{z^a}{(z^2+1)^2} dz \right|$

$\begin{cases} |z^a| = \rho^a \\ |z^2+1| \geq 1-\rho^2 \end{cases}$ for $z \in C_\rho$ with $0 < \rho < 1$

$$\Rightarrow \left| \int_{C_\rho} f(z)\, dz \right| \leq \frac{\rho^a}{(1-\rho^2)^2} \cdot \pi \rho$$

$$= \frac{\pi \rho^{1+a}}{(1-\rho^2)^2} \to 0 \text{ as } \rho \to 0$$

(To be cont'd)

$(-1 < a < 3)$