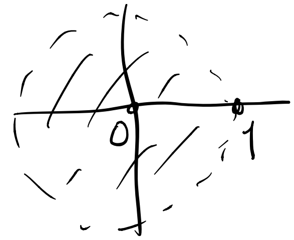
eg3: $f(z) = \dfrac{1}{z^2(1-z)}$ $\quad 0<|z|<1$



$$= \frac{1}{z^2}\left(\frac{1}{1-z}\right)$$

$$= \frac{1}{z^2}\left(1+z+z^2+\cdots\right)$$

$$= \underbrace{\frac{1}{z^2}+\frac{1}{z}}_{\text{principal part has only finite many terms.}}+1+z+\cdots$$

$\therefore$ $z=0$ is a pole of order 2 of $f(z)$.

& $z^2 f(z) = \dfrac{1}{1-z}$ which is analytic in $\{|z|<1\}$.

eg4: $f(z) = \dfrac{z^2+z-2}{z+1} = \underbrace{-\dfrac{2}{z+1}}_{\text{principal part}} - 1 + (z+1)$

$\Rightarrow$ $z=-1$ is a simple pole of $f(z)$.

§80 Residues at Poles

Thm: Let z_0 be an isolated singular point of a function $f(z)$. Then the followings are equivalent:

(a) z_0 is a pole of order m $(m=1,2,\cdots)$ of $f(z)$.

(b) $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0

Moreover, if (a) & (b) are ture, then

$$\operatorname*{Res}_{z=z_0} f(z) = \begin{cases} \phi(z_0) & \text{if } m=1 \\ \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}, & \text{otherwise} \end{cases}$$

Pf: By Note 2

$$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$(b_m \neq 0)$$

$$\Leftrightarrow (z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1(z-z_0)^{m-1}$$
$$+ a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \cdots$$

$= \phi(z)$ analytic at z_0

with $\phi(z_0) = b_m \neq 0$

This proved (a) $\Leftrightarrow$ (b).

And

$$\operatorname{Res}_{z=z_0} f(z) \stackrel{def}{=} b_1 = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Note that if $m=1$, $\dfrac{\phi^{(m-1)}(z_0)}{(m-1)!} = \dfrac{\phi(z_0)}{0!} = \phi(z_0)$.

#

§81 Examples

eg 2: If $f(z) = \dfrac{z^3+2z}{(z-i)^3}$

$z=i$ is an isolated singular.

$\phi(z) = z^3+2z$ is analytic (at i) &

$\phi(i) = i^3+2i = i \neq 0$

$\Rightarrow$ $z=i$ is a pole of order 3 of $f(z)$

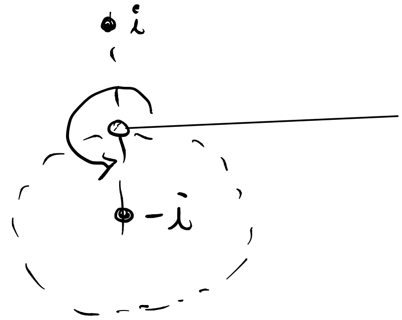
And $\operatorname{Res}_{z=i} f(z) = \dfrac{\phi^{(2)}(i)}{2!} = \dfrac{(6z)\big|_{z=i}}{2}$

$= 3i$

eg3: Let $f(z) = \dfrac{(\log z)^3}{z^2+1}$ where the branch of log is $\log z = \ln r + i\theta$, $0 < \theta < 2\pi$

Since this branch of log is analytic in

$$|z+i| < 1,$$



$f(z)$ is analytic in

$$0 < |z+i| < 1$$

and has an isolated singular point at $z=-i$.

$$f(z) = \frac{(\log z)^3}{(z+i)(z-i)} = \frac{\left(\frac{(\log z)^3}{z-i}\right)}{z-(-i)}$$

Since $\phi(z) = \dfrac{(\log z)^3}{z-i}$ is analytic in $|z+i|<1$

and $\phi(-i) = \dfrac{(\log(-i))^3}{-i-i} = \dfrac{\left(i\frac{3\pi}{2}\right)^3}{-2i} = \dfrac{27\pi^3}{16} \neq 0$

$\Rightarrow$ $z=-i$ is a simple pole of $f(z)$ and

$$\operatorname{Res}_{z=-i} f(z) = \phi(-i) = \frac{27\pi^3}{16}.$$

eg4 $f(z) = \dfrac{1-\cos z}{z^3}$

Note that $(1-\cos z)\big|_{z=0} = 0$

$\therefore$ $z=0$ is not a pole of order 3

In deed

$$f(z) = \frac{1-\cos z}{z^3} = \frac{1}{z^3}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\right]$$

$$= \frac{1}{z^3}\left[\frac{z^2}{2!}-\frac{z^4}{4!}+\cdots\right]$$

$$= \frac{1}{2z} - \frac{z}{4!} + \cdots$$

$\Rightarrow$ $z=0$ is a simple pole of $\dfrac{1-\cos z}{z^3}$

& $\operatorname{Res}_{z=0} \dfrac{1-\cos z}{z^3} = \dfrac{1}{2}$. ※

§82 Zeros of Analytic Functions

Def: Suppose f is analytic at z_0. If there is a positive integer $m \geq 1$ such that

$$\begin{cases} f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \text{ and} \\ f^{(m)}(z_0) \neq 0 \end{cases}$$

then f is said to have a zero of order m at z_0.

Thm 1: Let f be analytic at z_0. Then $f(z)$ has a zero of order m at z_0 if and only if there is an analytic function $g(z)$ such that

$g(z_0) \neq 0$ and

$$f(z) = (z-z_0)^m g(z)$$

Pf: f has a zero of order m at z_0

$$\Leftrightarrow \quad f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$$

$$\Leftrightarrow \quad f(z) = (z-z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots \right]$$

$$= (z-z_0)^m g(z)$$

where $g(z) = \begin{cases} \dfrac{f^{(m)}(z_0)}{m!} + \dfrac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots, & z \neq z_0 \\ \dfrac{f^{(m)}(z_0)}{m!} \neq 0, & z = z_0 \end{cases}$

analytic

※

<u>Thm 2</u>: Suppose f is analytic and z_0 is a zero of f, but $f(z) \not\equiv 0$ in any neighborhood of z_0. Then $\exists\, \varepsilon > 0$ such that

$$f(z) \neq 0 \quad \forall\ 0 < |z-z_0| < \varepsilon.$$

(i.e. z_0 is the only zero of f in the disk $\{|z-z_0| < \varepsilon\}$ in other words, zeros of f are isolated.)

Pf: If $f(z) \not\equiv 0$ in any neighborhood of z_0

$\Rightarrow$ Taylor's expansion of f about $z_0 \not\equiv 0$

$\Rightarrow$ z_0 is a zero of order m for some finite $m \geq 1$.

$\Rightarrow$ $f(z) = (z-z_0)^m g(z)$, g analytic & $g(z_0) \neq 0$.

$\Rightarrow$ $\exists\, \varepsilon > 0$ s.t. $g(z) \neq 0$ $\forall\, z \in \{|z-z_0| < \varepsilon\}$.

$\Rightarrow \quad f(z) = (z-z_0)^m g(z) \neq 0, \ \forall \ 0 < |z-z_0| < \varepsilon.$

\#

<u>Thm3</u>: Suppose that f is analytic in a neighborhood N_0 of z_0 and $f(z)=0$ at each point z of a domain or line segment containing z_0. Then $f(z) \equiv 0$ in N_0.

<u>Pf</u>: $f(z)=0$ in a domain & line segment

$\Rightarrow \quad z_0$ is not isolated.

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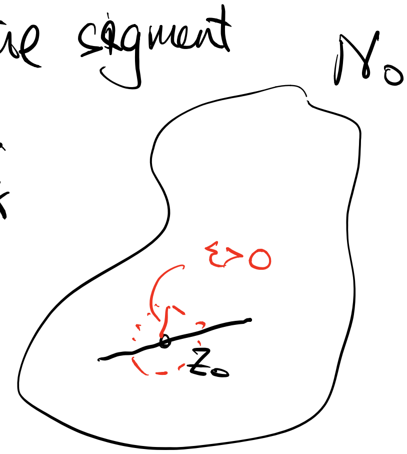

<u>§83 Zeros and Poles</u>

<u>Thm</u> Suppose that

(a) $p(z)$ and $q(z)$ are analytic at a point z_0.

(b) $p(z_0) \neq 0$ and $q(z)$ has a zero of order m at z_0.

Then $f(z) = \dfrac{p(z)}{q(z)}$ has a pole of order m at z_0.

Pf: By (b) $\Rightarrow$ $q(z) = (z-z_0)^m h(z)$

with $h(z_0) \neq 0$

h analytic

$$\Rightarrow f(z) = \frac{p(z)}{(z-z_0)^m h(z)} = \frac{\left(\frac{p(z)}{h(z)}\right)}{(z-z_0)^m}$$

with $\phi(z) = \dfrac{p(z)}{h(z)}$ analytic &

$$\phi(z_0) = \frac{p(z_0)}{h(z_0)} \neq 0 .$$ ⨳