<u>Def</u>: Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges $\forall z$ in a region Ω with sum $S(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. If $\forall \varepsilon>0$, $\exists N_\varepsilon>0$, <u>independent of $z\in\Omega$</u>, such that $\forall z\in\Omega$,

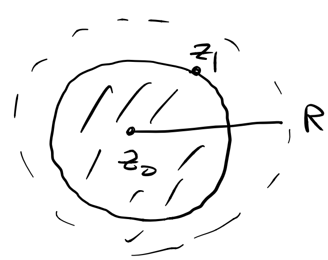
$$|\rho_N(z)| = |S(z)-S_N(z)| < \varepsilon, \quad \forall N>N_\varepsilon.$$

Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is said to <u>uniformly convergent</u> in the region Ω.

(or the convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is uniform in Ω)

(S_N is the N-partial sum).

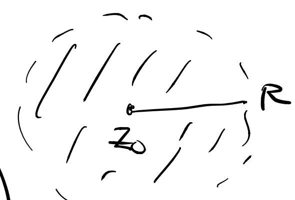
<u>Thm 2</u> If z_1 is a point inside the circle of convergence $|z-z_0|=R\,(>0)$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then the series must be uniformly convergent in the closed disk $|z-z_0|\le|z_1-z_0|$.



(Pf: Omitted)

§70 Continuity of Sums of Power Series

Thm A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z-z_0|=R$.



(Pf: Use uniform convergence on every closed disk $|z-z_0| \le R_1 < R$. Details omitted.)

Note: One can modify Thms 1 & 2 in §69 to conclude that if a Laurent Series expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n \quad \text{is valid in } R_1 < |z-z_0| < R_2,$$

then the series converges absolutely and uniformly in any $r_1 \le |z-z_0| \le r_2$ with $R_1 < r_1 < r_2 < R_2$;

and hence "Laurent series" represents a continuous function in $R_1 < |z-z_0| < R_2$.

§71 Integration and Differentiation of Power Series

Thm 1 Let C be a contour interior to the circle of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, and $g(z)$ be any function that is continuous on C. Then

$$\boxed{\int_C g(z)\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz}$$

(i.e. $\sum_{n=0}^{\infty} a_n g(z)(z-z_0)^n$ can be integrated term-by-term!)

Cor: The sum $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic at each point z interior to the circle of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Pf: In Thm 1, take $g(z) \equiv 1$,

and $\int_C (z-z_0)^n dz = 0$, $\forall n = 0, 1, 2, \cdots$

for all closed contour C. (interior to the circle of convergence)

Then Thm 1 $\Rightarrow$ $\int_C S(z)dz = 0$, $\forall$ closed contour C

Morera Thm (Thm 2 of §57) $\Rightarrow$ $S(z)$ is analytic. ※

eg1: Show that $f(z) = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$

is an entire function.

Pf: Since $\sin z = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!}$ valid $|z| < \infty$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$\Rightarrow$ if $z \neq 0$, $f(z) = \dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots$

$0 < |z| < \infty$

if $z = 0$, the series

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = 1 - 0 + 0 - \cdots$$

$$= 1 = f(0)$$

$\therefore$ $f(z) = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots$, $\forall\ |z| < \infty$.

Then our corollary $\Rightarrow$ $f(z)$ is entire. #

Thm 2 The power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be differentiated term-by-term inside the circle of convergem. That is

$$\boxed{\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}}$$

$$\left(S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \right)$$

$\forall$ point inside the circle of convergence.

eg: For $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $\quad |z|<\infty$

then Thm2 $\Rightarrow$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \cdot (2n+1) \frac{z^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad (|z|<\infty)$$

§72 Uniqueness of Series Representation

Thm1: If a series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to a $f(z)$ in $\{|z-z_0|<R\}$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the Taylor's series of $f(z)$ about z_0,

i.e. $\quad a_n = \frac{f^{(n)}(z_0)}{n!}$, $\forall n=0,1,2,\cdots$.

Pf : By Thm 2, we can differentiate term-by-term

$$\Rightarrow \quad f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n (z-z_0)^{n-2}$$

$$\vdots$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z-z_0)^{n-k}$$

$$\Rightarrow \quad f^{(k)}(z_0) = k(k-1)\cdots 1 \cdot a_k \quad \left(\begin{array}{l} n=k \text{ is the constant,} \\ \text{all other term } = 0 \end{array}\right)$$

$$= k!\, a_k$$

※

Similarly, one has

Thm 2 If $\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ converges to $f(z)$ in $R_1 < |z-z_0| < R_2$, then $\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ is the Laurent series of f about z_0.

(Pf : Omitted)

§73 Multiplication and Division of Power Series

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$

$$\Rightarrow \boxed{f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z-z_0)^n .}$$

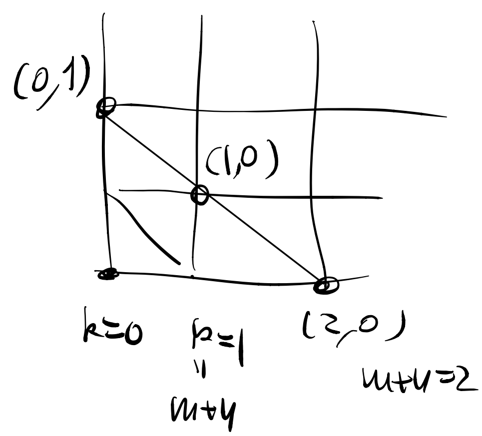
Pf: $f(z)g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \sum_{m=0}^{\infty} b_m (z-z_0)^m$

$$= \sum_{n,m=0}^{\infty} (a_n a_m)(z-z_0)^{m+n}$$

Let $k = m+n$,

$$= \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell} \right) (z-z_0)^k$$

$\divideontimes$



Another way to see this is by <u>Leibniz's rule</u>

$$\left(f(z)g(z) \right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) .$$