<u>eg1</u>: let $f(z) = \frac{1}{1-z}$

(i) Check that $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = 1 + z + z^2 + \cdots \quad (|z|<1)$

(Ex!)

(ii) Note that $|z|<1$, then $|-z|<1$,

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 + (-z) + (-z)^2 + \cdots \quad (|z|<1)$$

$$= 1 - z + z^2 - \cdots$$

(Ex: Check that this is the Taylor series expansion for $\frac{1}{1+z}$ about $z_0 = 0$ by calculating
$\frac{d^n}{dz^n}\left(\frac{1}{1+z}\right)\Big|_{z=0}$.)

(iii) let $\zeta = 1 - z$, then $|\zeta - 1| = |z| < 1$

$$\frac{1}{\zeta} = \frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots \quad (|z|<1)$$

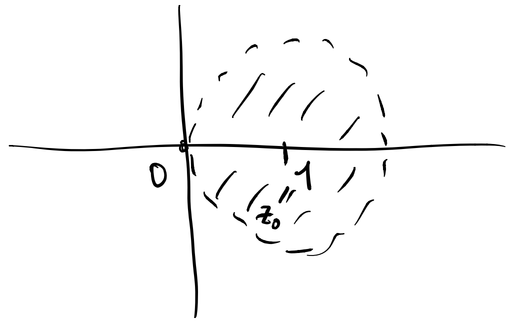
$$= 1 + (1-\zeta) + (1-\zeta)^2 + \cdots + (1-\zeta)^n + \cdots \quad (|\zeta - 1| < 1)$$

$$= 1 - (\zeta - 1) + (\zeta - 1)^2 + \cdots + (-1)^n (\zeta - 1)^n + \cdots$$

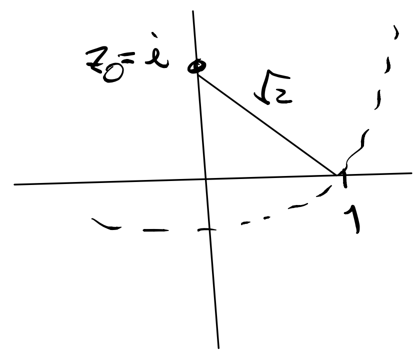
$$= \sum_{n=0}^{\infty} (-1)^n (\zeta - 1)^n \quad (|\zeta - 1| < 1)$$

Replace ζ by z, we have $\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$, $|z-1|<1$

is the Taylor's series expansion for $\frac{1}{z}$ about $z_0 = 1$



(iv) $f(z) = \dfrac{1}{1-z}$ is analytic at $z_0 = i$



In fact, f is analytic in $|z - i| < \sqrt{2}$

$$f(z) = \frac{1}{1-z} = \frac{1}{(1-i) - (z-i)}$$

$$= \frac{1}{1-i} \cdot \frac{1}{1 - \left(\frac{z-i}{1-i}\right)}$$

Since $|z - i| < \sqrt{2}$, $\left|\dfrac{z-i}{1-i}\right| = \dfrac{|z-i|}{\sqrt{2}} < 1$

$$\Rightarrow f(z) = \frac{1}{1-i} \cdot \left(1 + \left(\frac{z-i}{1-i}\right) + \left(\frac{z-i}{1-i}\right)^2 + \cdots \right)$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i} \right)^n , \qquad |z-i| < \sqrt{2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}} (z-i)^n , \qquad |z-i| < \sqrt{2} .$$

(Check: this is the Taylor series of $\frac{1}{1-z}$ about i)

<u>eg 2</u> (Easy) $f(z) = z^3 e^{2z}$ (about $z_0 = 0$)

(Ex!)

$$z^3 e^{2z} = z^3 \left(\sum_{n=0}^{\infty} \frac{1}{n!} (2z)^n \right)$$

$$= \sum_{\cancel{n}\,k-3=0}^{\infty} \frac{2^{\cancel{n}\,k-3}}{\cancel{n}\,(k-3)!} z^{\cancel{n+3}\,k} \qquad (n+3=k)$$

$$= \sum_{k=3}^{\infty} \frac{2^{k-3}}{(k-3)!} z^k , \qquad |z| < \infty$$

<u>eg 3</u> $$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \frac{1}{2i} \left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right]$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{i^n - (-i)^n}{n!} z^n$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{1-(-1)^n}{n!} i^n z^n \qquad 1-(-1)^n = \begin{cases} 0 & \text{even} \\ 2 & \text{odd} \end{cases}$$

$$= \frac{1}{2i} \sum_{k=0}^{\infty} \frac{2}{(2k+1)!} (i)^{2k+1} z^{2k+1} \quad \left(\begin{matrix} n = \text{odd} = 2k+1 \\ k = 0, 1, 2, \ldots \end{matrix} \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} i^{2k} z^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \qquad (|z| < \infty)$$

Reading exercise: egs 4, 5, 6 in the text book.

<u>Note of eg6</u>: $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad (|z| < \infty)$

$$\Rightarrow \cosh z = \cosh(z - 2\pi i) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (z - 2\pi i)^{n}$$

is the Taylor's series expansion of $\cosh z$ about $z_0 = 2\pi i$. $\quad (|z - 2\pi i| < \infty)$

§65 Negative Powers of $z - z_0$

eg2: $\cosh\left(\frac{1}{z}\right) \quad \left(\text{for } 0 < |z| < \infty \Leftrightarrow 0 < \frac{1}{|z|} < \infty \right)$

using the Taylor's expansion of $\cosh z$ $\rightarrow$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{1}{z^{2n}}$$

$$= 1 + \frac{1}{2!\, z^2} + \frac{1}{4!\, z^4} + \cdots \quad (0 < |z| < \infty)$$

<u>eg3</u>: Expand $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ in power of z $(0 < |z| < 1)$

<u>Soln</u>: $f(z) = \dfrac{1+2z^2}{z^3+z^5} = \dfrac{1+2z^2}{z^3(1+z^2)}$

$$= \frac{1}{z^3} \cdot \frac{1+2z^2}{1+z^2}$$

$$= \frac{1}{z^3} \cdot \frac{1-2+2(1+z^2)}{1+z^2}$$

$$= \frac{1}{z^3} \cdot \left[2 - \frac{1}{1+z^2} \right]$$

$$= \frac{1}{z^3} \left[2 - \sum_{n=0}^{\infty} (-1)^n (z^2)^n \right]$$

since $|z|<1$ $\Downarrow$ $|z^2|<1$

$$= \frac{1}{z^3} \left[2 - (1 - z^2 + z^4 - \cdots) \right]$$

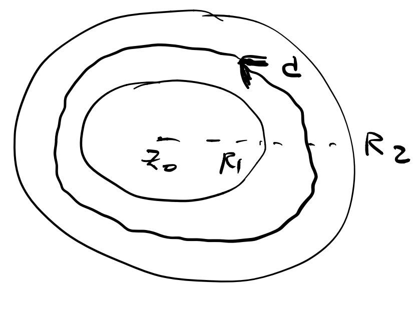
$$= \frac{1}{z^3} \left(1 + z^2 - z^4 + z^6 - \cdots \right) \quad (|z|<1)$$

$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \cdots \quad (|z|<1)$$

§66 Laurent Series

Thm (Laurent) Suppose that a function f is analytic throughout an annulus domain $R_1 < |z - z_0| < R_2$, and C denote any positively oriented simple closed contour around z_0 and lying in the domain

Then at each point z in the domain, $f(z)$ has the series representation



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2)$$

where

$$\begin{cases} a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(s)\,ds}{(s - z_0)^{n+1}}, & n = 0, 1, 2, \cdots \\ b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(s)\,ds}{(s - z_0)^{-n+1}}, & n = 1, 2, 3, \cdots \end{cases}$$

Let $c_n = \begin{cases} a_n & \text{if } n \geq 0 \\ b_{-n} & \text{if } n < 0 \end{cases}$, Then

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad (R_1 < |z - z_0| < R_2)}$$

where $\quad c_n = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z_0)^{n+1}} \quad (n=0,\pm 1,\pm 2,\dots)$

Notes: (i) Both forms are called a Laurent Series expansion (or representation) of $f(z)$.

(ii) The Theorem asserts that both series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \& \quad \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{converge}$$

for $z \in \{R_1 < |z-z_0| < R_2\}$ and their sum equals $f(z)$.

(iii) R_1 could be zero ; R_2 could be infinite

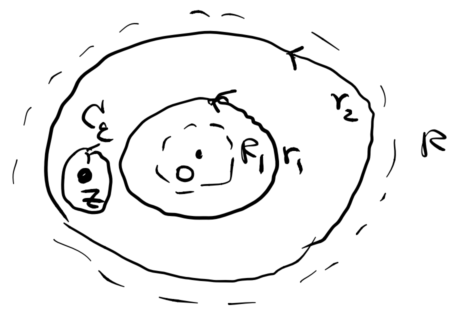
we may have
$$\begin{cases} 0 < R_1 < |z-z_0| < R_2 < \infty \\ 0 < |z-z_0| < R_2 < \infty \\ 0 < R_1 < |z-z_0| < \infty \\ 0 < |z-z_0| < \infty \end{cases}$$

(iv) In case that f is actually analytic in $|z-z_0| < R_2$. Then one can show that $b_n = 0, \forall n = 1,2,\cdots$ and the Laurent series becomes the Taylor's series about z_0. (Ex!)

§67 Proof of Laurent's Theorem

(Sketch:)

Case $z_0 = 0$



Consider $\{r_1 \le |z| \le r_2\}$

with $R_1 < r_1 < r_2 < R_2$.

Let $C_1 = \{|z| = r_1\}$ & $C_2 = \{|z| = r_2\}$

Then f is analytic on C_1 & C_2 and between them.

Let $z \in \{r_1 < |z| < r_2\}$

Then $\exists \varepsilon > 0$ such that

$$B_\varepsilon(z) \subset \{r_1 < |z| < r_2\}$$

Let $C_\varepsilon = \partial B_\varepsilon(z) = \{|s - z| = \varepsilon\}$

Applying Cauchy-Goursat Thm to the analytic function $\dfrac{f(s)}{s-z}$, we have

$$\int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)ds}{s-z} - \int_{C_\varepsilon} \frac{f(s)ds}{s-z} = 0$$

Then Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(s)}{s-z}\,ds$$

$$= \frac{1}{2\pi i}\left[\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)}{s-z}\,ds\right]$$

For $s \in C_2$, $\left|\frac{z}{s}\right| = \frac{|z|}{r_2} < 1$

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s}\cdot\frac{1}{1-\frac{z}{s}}\,ds$$

$$= \sum_{n=0}^{\infty} a_n z^n \quad (\text{Ex!})$$

as the remainder

$$\left|\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds\right| \to 0 \text{ as } N \to \infty.$$

(Ex!)

For $s \in C_1$, $\left|\frac{z}{s}\right| = \frac{|z|}{r_1} > 1$ (ie. $\left|\frac{s}{z}\right| < 1$)

Hence $$\frac{-1}{2\pi i}\int_{C_1} \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z}\cdot\frac{1}{1-\frac{s}{z}}\,ds$$

(Check)
$$= \sum_{\substack{\cancel{n=0}\\ k=1}}^{N\ \cancel{N-1}} \left(\frac{1}{2\pi i}\int_{C_1} f(s)\, s^{\cancel{n}\,k-1}\, ds\right) \frac{1}{z^{\cancel{n+1}\,k}}$$
$$+ \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$
$$= \sum_{k=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)\,ds}{s^{-k+1}}\right)\frac{1}{z^k}$$
$$+ \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

(Check):
$$\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds\right| \le \frac{M_1 r}{r-r_1}\left(\frac{r_1}{r}\right)^N \xrightarrow[\text{as } N\to+\infty]{} 0$$

$$\left(\text{where} \quad M_1 = \sup_{|s|=r_1} |f(s)|, \quad r = |z| > r_1\right)$$

This completes the proof of Laurent Theorem.
(for case $z_0=0$) (Using deformation of paths)
General case follows easily.

#

§68 Examples

eg1: Find Laurent series expansion of

$$f(z) = \frac{1}{z(1+z^2)} \quad \text{on} \quad 0<|z|<1$$

(Note: f has "singularities" at $z=0, \pm i$.
$\therefore$ f is analytic on $\{0<|z|<1\}$)

Soln: $f(z) = \dfrac{1}{z(1+z^2)}$ $\quad (0<|z|<1 \Rightarrow 0<|z^2|<1)$

$$= \frac{1}{z} \cdot \frac{1}{1-(-z^2)}$$

$$= \frac{1}{z}\left(1+(-z^2)+(-z^2)^2+\cdots\right)$$

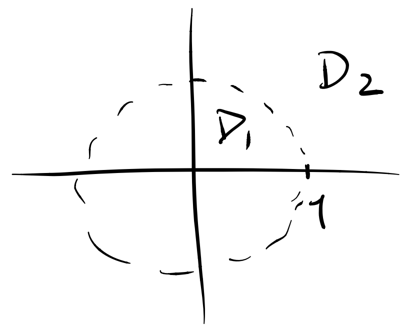
$$= \frac{1}{z}\left(1-z^2+z^4-\cdots\right)$$

$$= \frac{1}{z} - z + z^3 - \cdots \quad (0<|z|<1)$$

$$\left(\text{general term}\right) = \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \quad (0<|z|<1)$$

eg 2: $f(z) = \frac{z+1}{z-1}$



analytic in $D_1 = \{|z|<1\}$

$D_2 = \{1<|z|<\infty\}$

On $D_1 = \{|z|<1\}$, $f(z) = \frac{z+1}{z-1}$ has a Taylor's series expansion

$$f(z) = -1 - 2\sum_{n=1}^{\infty} z^n \quad (|z|<1)$$

(Ex!)

On $D_2 = \{1<|z|<\infty\}$, we have

$$f(z) = \frac{z+1}{z-1} = \frac{z+1}{z} \cdot \frac{1}{1-\frac{1}{z}} \qquad \left(\begin{array}{l} |z|>1 \\ \Rightarrow \frac{1}{|z|}<1 \end{array} \right)$$

$$= \left(1+\frac{1}{z}\right)\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$= 1+\frac{1}{z}+\frac{1}{z^2}+\cdots$$

$$+\frac{1}{z}+\frac{1}{z^2}+\cdots$$

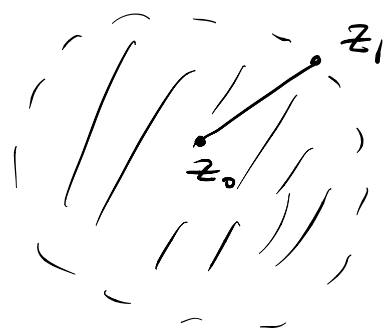
$$= 1+\frac{2}{z}+\frac{2}{z^2}+\frac{2}{z^3}+\cdots \quad (1<|z|<\infty)$$

(Ex: find the general term)

Collection of General Facts of Power Series

§69 Absolute and Uniform Convergence of Power Series

Thm 1: If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < |z_1 - z_0|$.



(Pf: Omitted)

Def: The greatest circle centered at z_0 such that the series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at each point inside is called the circle of convergence of the series

Cor: For any z_2 outside the circle of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, the series $\sum_{n=0}^{\infty} a_n (z_2 - z_0)^n$ diverges.

(Pf: Omitted)