<u>Def</u>: A series $\sum_{n=1}^{\infty} z_n$ is said to be <u>absolutely convergent</u> if the (real) series $\sum_{n=1}^{\infty} |z_n|$ converges.

<u>Cor 2</u>: The absolute convergence of a series of <u>complex</u> numbers implies the convergence of that series.

<u>Pf</u>: $\sum_{n=1}^{\infty} z_n$ absolutely convergence

$\Leftrightarrow \sum_{n=1}^{\infty} |z_n|$ converges.

By $|x_n| \leq |z_n|$, $|y_n| \leq |z_n|$,

$\sum_{n=1}^{\infty} |x_n|$ & $\sum_{n=1}^{\infty} |y_n|$ converge by comparison test.

Hence $\sum_{n=1}^{\infty} x_n$ & $\sum_{n=1}^{\infty} y_n$ converge (in $\mathbb{R}$)

$\Rightarrow \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$ converges.

(by Thm).

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Terminology: For a series $\sum_{n=1}^{\infty} z_n = S$

with partial sum $S_N = \sum_{n=1}^{N} z_n$,

Then $\rho_N = S - S_N = \sum_{n=N+1}^{\infty} z_n$ is called

the remainder after N terms of the series.

By definite, $\sum_{n=1}^{\infty} z_n$ converges to S

$\iff$ $\rho_N = S - S_N \to 0$ as $N \to \infty$.

$(\& \ S = S_N + \rho_N)$

eg: $\forall z$ such that $|z| < 1$. Then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Pf: We need to check

$$|\rho_N| = \Big| \underbrace{\frac{1}{1-z}}_{S(z)} - \underbrace{\sum_{n=0}^{N-1} z^n}_{S_N(z)} \Big| \to 0 \quad \text{as } N \to \infty$$

By straight forward calculation

$$S_N(z) = \sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z} \quad (\text{Ex!})$$

Hence

$$\rho_N(z) = S(z) - S_N(z)$$
$$= \frac{1}{1-z} - \frac{1-z^N}{1-z}$$
$$= \frac{z^N}{1-z}$$

$$\therefore \quad |\rho_N(z)| = \frac{|z|^N}{|1-z|} \to 0 \quad \text{as } N \to \infty$$

since $|z|<1$

$$\therefore \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \#$$

§62 Taylor Series

Thm (Taylor's Theorem)

Suppose that a function f is analytic throughout a disk $\{|z-z_0|<R_0\}$ $(R_0>0)$. Then $f(z)$ has the power series representation

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad |z-z_0|<R_0}$$

Taylor series expansion of $f(z)$ about the point z_0.

Notes: (i) f analytic at a point z_0

$\Rightarrow$ f analytic in a disk centered at z_0

$\Rightarrow$ f has a Taylor series expansion at z_0

(ii) If f is entire, then f is analytic in $|z-z_0|<R_0$

for any $R_0>0$.

$\Rightarrow$ the Taylor series expansion valid on $\mathbb{C}$.

(iii) No convergence test is needed as long as f

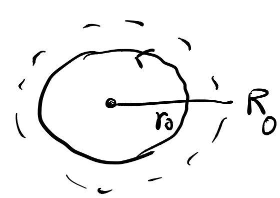
is analytic in $|z-z_0|<R_0$.

(iv) For $z_0=0$, then $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$

is called the Maclaurin Series.

§63 Proof of Taylor's Theorem

Pf: Special case $z_0=0$.



Let f analytic in $\{|z|<R_0\}$

Then $\forall\, r_0>0$ s.t. $0<r_0<R_0$,

f is analytic inside and on $C_0=\{|z|=r_0\}$

Cauchy integral famula $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)\, ds}{s-z} \qquad \forall \quad |z| = r < r_0$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s}\left[\frac{1}{1-(z/s)}\right] ds$$

$$\left(s \in C_0 \Leftrightarrow |s| = r_0 \quad \text{hence} \quad \left|\frac{z}{s}\right| = \frac{r}{r_0} < 1 \right)$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s}\left[1 + \left(\frac{z}{s}\right) + \cdots + \left(\frac{z}{s}\right)^{N-1} + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right] ds$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s}\left[1 + \left(\frac{z}{s}\right) + \cdots + \left(\frac{z}{s}\right)^{N-1}\right] ds$$

$$+ \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s} \cdot \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}} ds$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s} \sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n \cdot ds$$

$$+ \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}} ds \right) z^n$$

$$+ \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \left(\frac{z}{s}\right)^N ds$$

$$= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n \qquad \text{(by Cauchy integral formula)}$$

$$+ \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \left(\frac{z}{s}\right)^N ds$$

$\therefore$

$$\rho_N(z) = f(z) - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \left(\frac{z}{s}\right)^N ds$$

Let $M_0 = \sup_{|s|=r_0} |f(s)|$

and note that $|s-z| \geq |s| - |z| = r_0 - r > 0$

& $\left|\frac{z}{s}\right| = \frac{r}{r_0} < 1$,

We have

$$|\rho_N(z)| \leq \frac{1}{2\pi} \cdot \frac{M_0}{r_0 - r} \left(\frac{r}{r_0}\right)^N \cdot 2\pi r_0$$

$$= \frac{r_0 M_0}{r_0 - r} \left(\frac{r}{r_0}\right)^N \to 0 \quad \text{as } N \to \infty$$

as $\frac{r}{r_0} < 1$.

$\therefore \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n .$

Finally, for general z_0 :

consider $g(z) = f(z + z_0)$ analytic in $|z| < R_0$

(Ex!) $\divideontimes$

§64 Examples

Need to remember all 6 expansions below:

- $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z|<1)$
- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \qquad (|z|<\infty)$
- $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad (|z|<\infty)$
- $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad (|z|<\infty)$
- $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \quad (|z|<\infty)$

- $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \quad (|z| < \infty)$