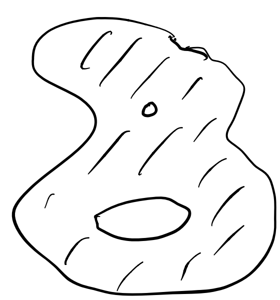
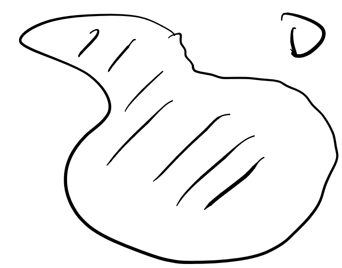
§52 Simply Connected Domains

Def: A <u>simply connected domain</u> D is a domain such that every simply closed contour within it encloses only points of D.

(i.e. domain that has no hole inside)

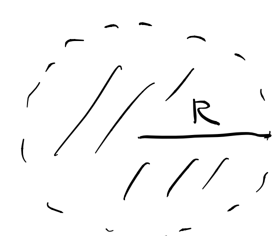


not simply connected



simply connected

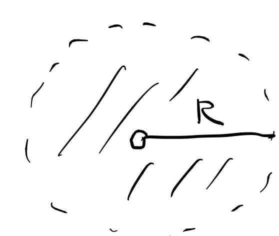
eg: • $D = \{z \in \mathbb{C} : |z| < R\}$ is simply connected



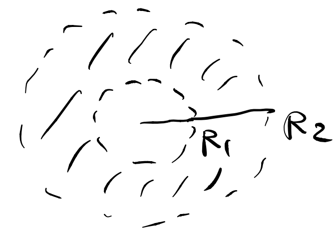
• $D \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < R\}$ is <u>not</u> simply-connected:

$\exists\ \gamma$ (says for eg circle of radius $0 < r < R$)

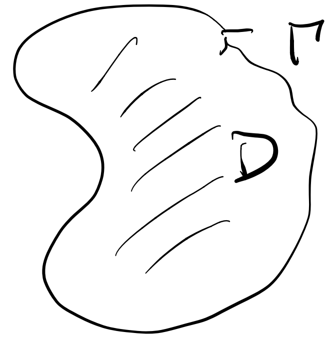
encloses a region that containing $z = 0 \notin D \setminus \{0\}$



- Similarly, any <u>annulus</u> $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$ $(0 < R_1 < R_2)$ is <u>not</u> simply connected



Typical situation in our course for simply-connected domains are domains bounded by a simply closed contour Γ.
(by Jordan Curve Thm)

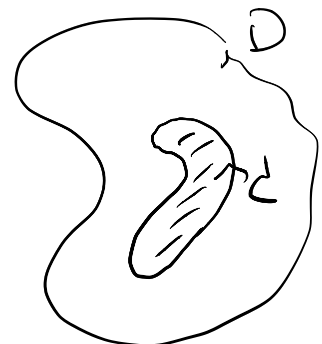


<u>Thm</u>: If a function f is <u>analytic</u> throughout a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

for any <u>closed</u> contour (not necessary simple) C lying in D.

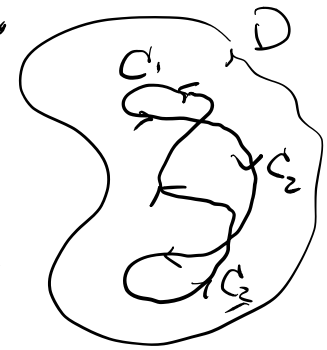
Pf: If C is simple, closed then the region enclosed by C is contained in D.



$\Rightarrow$ f is analytic interior to and on C.

$\therefore$ Cauchy-Goursat Thm $\Rightarrow$ $\int_C f(z)dz = 0$

If C is not simple, just closed, but intersects itself a finite number of times.



Then C can be subdivided into finitely many simple closed contours C_i lying in D.

Then $\int_C f(z)dz = \sum_i (\pm) \int_{C_i} f(z)dz = 0$

as $\int_{C_i} f(z) = 0$, $\forall i$

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(Infinitely many intersection: Omitted)

eg: $D = \{ |z| < 2 \}$ simply-connected.

$$f(z) = \frac{\sin z}{(z^2+9)^5} \quad \text{is analytic in } D$$

(since the singularities are $z = \pm 3i \notin D$)

$\therefore$ By the thm, $\displaystyle\int_C \frac{\sin z}{(z^2+9)^5}\,dz = 0$

$\forall$ closed contour in $\{ |z| < 2 \}$.

Cor 1: A function f that is analytic throughout a simply-connected domain D must have an antiderivative everywhere in D.

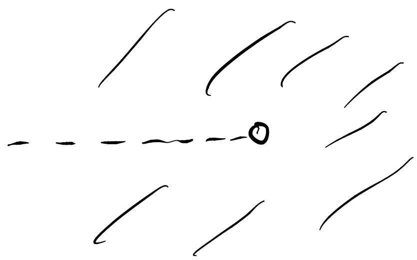
(Pf: By the Thm in §48)

Cor 2: Entire functions always possess antiderivatives.

(Pf: $\mathbb{C}$ is simple connected.)

<u>Note:</u> Compare : principal branch of $\operatorname{Log} z$

and principal value of $\operatorname{Log} z$.

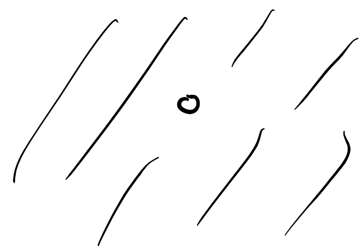


$D = \{ r > 0, -\pi < \theta < \pi \}$

is simple connected

$\frac{1}{z}$ analytic on D

$\Rightarrow$ antiderivative exists which is the principal branch of $\operatorname{Log} z$ (up to a constant.)



$\mathbb{C} \setminus \{0\}$

is not simple connected

even $\frac{1}{z}$ analytic on $\mathbb{C} \setminus \{0\}$

it has no antiderivative on the whole $\mathbb{C} \setminus \{0\}$

(Principal Value on $\mathbb{C} \setminus \{0\}$ is not continuous.)

§53 Multiply Connected Domains

Def: A domain that is not simply-connected is said to be multiple connected.

Thm: Suppose that

(a) C is a simple closed contour in counterclockwise direction

(b) C_k, $k = 1, 2, \cdots, n$ are simple closed contours interior to C in clockwise direction, they are disjoint and whose interiors are also disjoint.

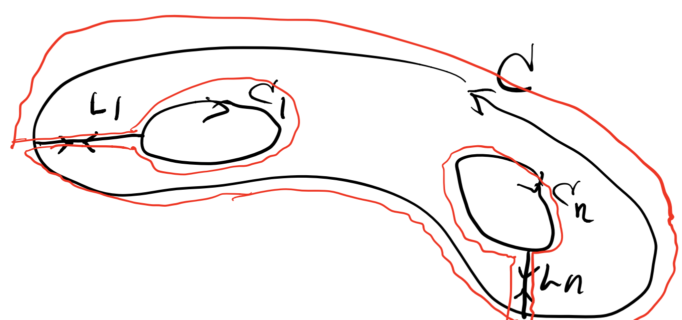
If a function f is analytic on C and C_k, $k=1,\cdots,n$ and throughout the (multiple connected) domain consisting of the points interior to C but exterior to C_k, $k=1,\cdots,n$, then

$$\boxed{\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.}$$

(Note: $\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{-C_k} f(z)\,dz$

counterclockwise direction.)

Pf:



Let L_i be polygonal path joining C to C_k, $k=1,\cdots,n$ in the multiple connected domain such that L_k has no self-intersection and L_k are disjoint.

Then a "simple closed contour" Γ can be formed

$$\Gamma = \text{part of } C + L_1 + C_1 + (-L_1) + \text{another part of } C + L_2 + C_2 + (-L_2) + \cdots + L_n + C_n + (-L_n)$$

+ final part of C.

$$= \text{"}C\text{"} + L_1 + C_1 - L_1 + L_2 + C_2 - L_2 + \cdots + L_n + C_n - L_n .$$

By Cauchy-Goursat Thm,

$$0 = \int_{\Gamma} f(z)\,dz = \left(\int_{C} + \int_{L_1} + \int_{C_1} - \int_{L_1} + \cdots + \int_{L_n} + \int_{C_n} - \int_{L_n}\right) f(z)\,dz$$

$$= \int_{C} f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

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