eg2 Let $C: z = z(t)$, $a \le t \le b$, $z_1 = z(a)$ & $z_2 = z(b)$.

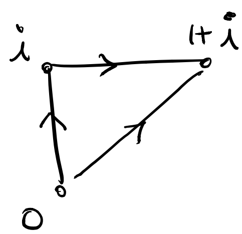
Then
$$\int_C z\,dz = \int_a^b z(t)\, d(z(t))$$
$$= \int_a^b z(t) z'(t)\,dt$$
$$= \frac{1}{2}\int_a^b \left\{\frac{d}{dt}[z(t)]^2\right\} dt$$
$$= \frac{1}{2}\left[z(b)^2 - z(a)^2\right]$$
$$= \frac{1}{2}(z_2^2 - z_1^2)$$
depends only on the beginning & end point z_1 & z_2, but not C.

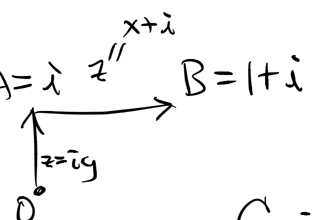
In this case, we write $\int_{z_1}^{z_2} z\,dz = \frac{1}{2}(z_2^2 - z_1^2)$.

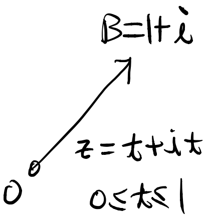
eg3: Let $f(z) = y - x - i3x^2$ for $z = x + iy$

(Note: $u = y - x$, $v = -3x^2$

$\Rightarrow$ $u_x = -1$, $u_y = 1$ $\not\equiv$ $v_x = -6x$, $v_y = 0$ not analytic)

Let 

C_1: 

C_2: 

$$\int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

$$= \int_0^1 f(iy)\,d(iy) \quad + \quad \int_0^1 f(x+i)\,d(x+i)$$

$$\left(\text{parametrization } \begin{array}{l} OA: z = iy, \\ 0 \le y \le 1 \end{array} \;\middle|\; \begin{array}{l} AB: z = x+i \\ 0 \le x \le 1 \end{array} \right)$$

$$= \int_0^1 y(i\,dy) + \int_0^1 (1-x-i3x^2)\,dx$$

$$= i\int_0^1 y\,dy + \int_0^1 (1-x)dx - 3i\int_0^1 x^2 dx$$

$$= \frac{1-i}{2} \quad (\text{check!})$$

$$\int_{C_2} f(z)dz = \int_0^1 f(t+it)\,d(t+it)$$

$$= \int_0^1 (-i3t^2)\big((1+i)dt\big)$$

$$= -3i(1+i)\int_0^1 t^2 dt$$

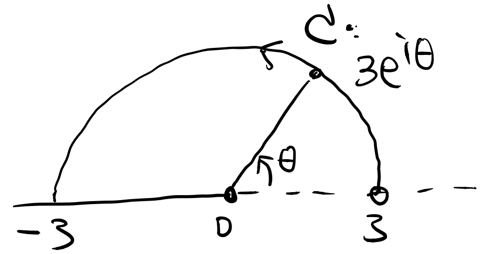
$$= 1-i \quad (\text{check})$$

$$\neq \frac{1-i}{2} = \int_{C_1} f(z)dz.$$

§46 Examples Involving Branch Cuts

eg1: Let C : $z = 3e^{i\theta}$, $0 \le \theta \le \pi$ (semicircular arc)

and $f(z) = z^{1/2}$.



Suppose we consider the following branch of $z^{1/2}$:

$$f(z) = z^{1/2} = \exp\left(\tfrac{1}{2}\log z\right), \quad |z|>0,\ \underline{\underline{0<\arg < 2\pi}}$$

(not Principal)

Then the initial point $z(0) = 3$ (of the arc C) doesn't belongs to the domain of this branch!

However, for this branch,

$$\begin{aligned} f(z(\theta))\, z'(\theta) &= z(\theta)^{1/2} z'(\theta) \\ &= \left(3e^{i\theta}\right)^{1/2} \cdot 3ie^{i\theta} \\ &= \sqrt{3}\, e^{i\frac{\theta}{2}} \cdot 3ie^{i\theta} \quad \text{for } 0<\theta\le\pi \\ &= 3\sqrt{3}\, i\, e^{i\frac{3\theta}{2}} \\ &\longrightarrow 3\sqrt{3}\, i \quad \text{as } \theta \to 0. \end{aligned}$$

$\Longrightarrow \quad \int_C f(z)\,dz$ exists.

And
$$\int_C f(z)\,dz = \lim_{\varepsilon\to 0}\int_\varepsilon^\pi f(3e^{i\theta})\, d(3e^{i\theta})$$
$$= \lim_{\varepsilon\to 0}\int_\varepsilon^\pi (3e^{i\theta})^{1/2}\, 3ie^{i\theta}\, d\theta$$
$$= \lim_{\varepsilon\to 0}\int_\varepsilon^\pi 3\sqrt{3}\, i e^{i\frac{3\theta}{2}}\, d\theta$$
$$= \lim_{\varepsilon\to 0}\left[2\sqrt{3}\, e^{i\frac{3\theta}{2}}\right]_\varepsilon^\pi$$
$$= \lim_{\varepsilon\to 0}\left[2\sqrt{3}\, e^{i\frac{3\pi}{2}} - 2\sqrt{3}\, e^{i\frac{3\varepsilon}{2}}\right]$$
$$= 2\sqrt{3}\,(e^{i\frac{3\pi}{2}} - 1)$$
$$= -2\sqrt{3}\,(1+i)$$

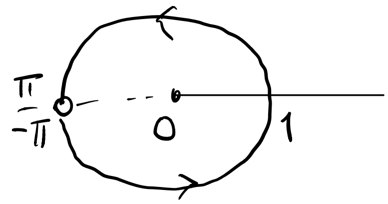
We usually simple write
$$\int_C z^{1/2}\,dz = \int_0^\pi 3\sqrt{3}\, i\, e^{i\frac{3}{2}\theta}\, d\theta = 2\sqrt{3}\left[e^{i\frac{3\theta}{2}}\right]_0^\pi$$

(even the 0 is not on the branch) (as in the real case)

---

eg: Evaluate $\int_C z^{-1+i}\,dz$ in principal branch along the unit circle C

Soln: Principal branch of



$$z^{-1+i} = \exp[(-1+i)\operatorname{Log} z]$$

$(-\pi < \operatorname{Arg} z < \pi)$

$$= \exp[(-1+i)(\ln|z| + i\operatorname{Arg} z)]$$

The unit circle C can be parametered as

$$C: \quad z = e^{i\theta}, \quad -\pi \le \theta \le \pi$$

(Then $\theta = \operatorname{Arg} z$ except at the end points)

$$\therefore \int_C z^{-1+i}\,dz = \int_{-\pi}^{\pi} e^{(-1+i)i\theta}\, d(e^{i\theta})$$

(By the same argument as in eg 1)

$$= \int_{-\pi}^{\pi} e^{-\theta - i\theta}\, i e^{i\theta}\, d\theta$$

$$= i\int_{-\pi}^{\pi} e^{-\theta}\, d\theta = i\left[-e^{-\theta}\right]_{-\pi}^{\pi}$$

$$= i[-e^{-\pi} + e^{\pi}] = 2i\sinh\pi$$

#

§47 Upper Bounds for Moduli of Contour Integrals

Lemma: If $w(t)$ is a piecewise continuous cpx-valued function defined on $a \le t \le b$, then

$$\boxed{\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt}$$

Pf: If $\int_a^b w(t)\,dt = 0$, then we are done.

If $\int_a^b w(t)\,dt \ne 0$, then it can be written

as $$\int_a^b w(t)\,dt = r_0 e^{i\theta_0}$$

where $r_0 = \left|\int_a^b w(t)\,dt\right| > 0$, & $\theta_0 \in \mathbb{R}$.

Then $$r_0 = e^{-i\theta_0} \int_a^b w(t)\,dt$$
$$= \int_a^b e^{-i\theta_0} w(t)\,dt$$

$r_0 \in \mathbb{R} \Rightarrow$ $$r_0 = Re\left[\int_a^b e^{-i\theta_0} w(t)\,dt\right]$$
$$= \int_a^b Re\left[e^{-i\theta_0} w(t)\right] dt$$

$$\leq \int_a^b |e^{-i\theta_0} w(t)| dt$$

$$= \int_a^b |w(t)| dt. \quad \#$$