Def: Principal value of z^c, denoted by

$$\boxed{P.V.\, z^c \stackrel{\text{def}}{=} e^{c \log z}}$$

coincide with the principal branch of z^c, $|z|>0$, $-\pi < \operatorname{Arg} z < \pi$.

Finally, we may also define exponential function with base c by $(c \neq 0)$

$$c^z = e^{z \log c}, \quad \forall z \in \mathbb{C}$$

$$= e^{z[\ln|c| + i(\operatorname{Arg} c + 2k\pi)]}, \quad k \in \mathbb{Z}.$$

is multiple-valued.

But for any value of $\log c$, $c \neq 0$, is specified, then $c^z = e^{z \log c}$ is an entire (single-valued) function with

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z \log c} \log c$$

$$= c^z \log c \quad (\longleftarrow \log c \text{ is the specified value})$$

Note: $f(z) = z^c$ defined on $\mathbb{C} \setminus \{0\}$

and if $c = 0$, $f(z) = z^0 \equiv 1$ for all $z \in \mathbb{C} \setminus \{0\}$

and hence can be extended to the entire

function $f(z) = z^0 \equiv 1$ on the whole $\mathbb{C}$.

However, for $c \neq 0$, $g(z) = c^z$ defined $\forall z \in \mathbb{C}$

and $g(0) = c^0 = 1$.

But 0^z is not defined.

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§36 Examples

eg1: $i^i = e^{i \log i} = e^{i[\ln|i| + i(\operatorname{Arg} i + 2k\pi)]}$ $k \in \mathbb{Z}$

$$= e^{-(\frac{\pi}{2} + 2k\pi)}$$

$$= e^{-(2k+\frac{1}{2})\pi}, \quad k \in \mathbb{Z}.$$

$$\text{P.V. } i^i = e^{i[\ln|i| + i(\operatorname{Arg} i)]} \qquad (k=0)$$

$$= e^{-\frac{\pi}{2}}.$$

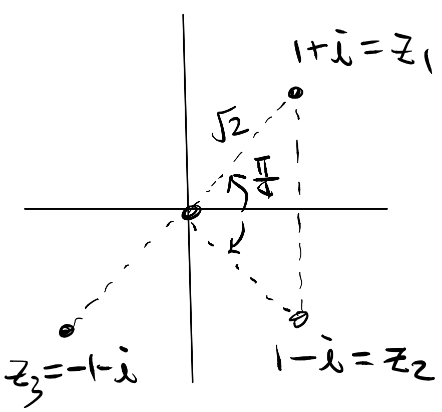
eg3 Principal branch of $z^{2/3}$

$$= e^{\frac{2}{3}\operatorname{Log} z} = e^{\frac{2}{3}(\ln r + i\Theta)}, \quad -\pi < \Theta < \pi, \ r > 0$$

$$= e^{\frac{2}{3}\ln r} e^{i\frac{2\Theta}{3}}$$

$$= \sqrt[3]{r^2}\left(\cos\frac{2\Theta}{3} + i\sin\frac{2\Theta}{3}\right), \quad -\pi < \Theta < \pi, \ r > 0$$

eg4: Let $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$



Then $P.V.\ z_1^i = e^{i\operatorname{Log} z_1}$

$$= e^{i[\ln\sqrt{2} + i\frac{\pi}{4}]}$$

$$= e^{-\frac{\pi}{4} + i\ln\sqrt{2}}$$

$$P.V\ z_2^i = e^{i\operatorname{Log} z_2}$$

$$= e^{i[\ln\sqrt{2} - i\frac{\pi}{4}]}$$

$$= e^{\frac{\pi}{4} + i\ln\sqrt{2}}$$

$$P.V.(z_1 z_2)^i = P.V.\ 2^i = P.V.\ e^{i\operatorname{Log} 2} = e^{i\ln 2}$$

$$(P.V.\ z_1^i)(P.V.\ z_2^i) = e^{-\frac{\pi}{4} + i\ln\sqrt{2}} e^{\frac{\pi}{4} + i\ln\sqrt{2}} = e^{i\ln 2}$$

$\therefore$ For z_1, z_2, $P.V.(z_1 z_2)^i = (P.V.\ z_1^i)(P.V.\ z_2^i)$.

(A good formula)

However, P.V. $z_3^i = e^{\frac{3\pi}{4} + i \ln\sqrt{2}}$ (Ex!)

$$\text{P.V. } (z_2 z_3)^i = e^{-\pi + i\ln 2} \quad \text{(Ex!)}$$

$$\neq (\text{P.V. } z_2^i)(\text{P.V. } z_3^i)$$

(The formula is not true for z_2, z_3!)

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§40 Inverse Trigonometric & Hyperbolic Functions

(1) $w = \sin^{-1} z$

<u>Soln</u>: $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

$$\Rightarrow \quad e^{iw} - 2iz - e^{-iw} = 0$$

$$\Rightarrow \quad (e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$$

$$\Rightarrow \quad e^{iw} = \frac{2iz + [(2iz)^2 + 4]^{1/2}}{2} \longleftarrow \text{multiple-valued}$$

$$= iz + (1 - z^2)^{1/2}$$

$$\Rightarrow \quad iw = \log[iz + (1-z^2)^{1/2}]$$

$$\Rightarrow \quad w = -i\log[iz + (1-z^2)^{1/2}]$$

$$\therefore \quad \boxed{\sin^{-1} z = -i \log[iz + (1-z^2)^{1/2}]}$$

multiple-valued.

Similarly

$$\boxed{\begin{array}{l} \cos^{-1} z = -i \log[z + i(1-z^2)^{1/2}] \\ \hline \tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z} \end{array}}$$

(Ex!)

eg: $\sin^{-1}(-i) = -i \log\left[i(-i) + (1-(-i)^2)^{1/2}\right]$

$$= -i \log\left[1 + 2^{1/2}\right]$$

$$= -i \log(1 \pm \sqrt{2})$$

$$= \begin{cases} -i\left[\ln(1+\sqrt{2}) + i\, 2k\pi\right], & k \in \mathbb{Z} \\ -i\left[\ln(\sqrt{2}-1) + i(2k+1)\pi\right], & k \in \mathbb{Z} \end{cases}$$

$$= \begin{cases} 2k\pi - i\ln(1+\sqrt{2}), & k \in \mathbb{Z} \\ (2k+1)\pi - i\ln(\sqrt{2}-1), & k \in \mathbb{Z} \end{cases}$$

$$= \begin{cases} n\pi - i\ln(1+\sqrt{2}), & n = \text{even} \\ n\pi - i\ln\frac{1}{1+\sqrt{2}}, & n = \text{odd} \end{cases}$$

$$= \begin{cases} n\pi - i\ln(1+\sqrt{2}), & n = \text{even} \\ n\pi + i\ln(1+\sqrt{2}), & n = \text{odd} \end{cases}$$

$$= n\pi + i(-1)^{n+1}\ln(1+\sqrt{2}), \quad n \in \mathbb{Z}$$

#

Derivatives (for branches)

- $\dfrac{d}{dz}\sin^{-1}z = \dfrac{1}{(1-z^2)^{1/2}}$
- $\dfrac{d}{dz}\cos^{-1}z = \dfrac{-1}{(1-z^2)^{1/2}}$

$\Big\}$ depend on the branch

- $\dfrac{d}{dz}\tan^{-1}z = \dfrac{1}{1+z^2}$ doesn't depend on the branch

(Ex!)

Inverse Hyperbolic Functions (multiple-valued)

$$\begin{cases} \sinh^{-1}z = \log\left[z+(z^2+1)^{1/2}\right] \\ \cosh^{-1}z = \log\left[z+(z^2-1)^{1/2}\right] \\ \tanh^{-1}z = \dfrac{1}{2}\log\dfrac{1+z}{1-z} \end{cases}$$

(Ex!)

with

- $\dfrac{d}{dz}\sinh^{-1}z = \dfrac{1}{(z^2+1)^{1/2}}$
- $\dfrac{d}{dz}\cosh^{-1}z = \dfrac{1}{(z^2-1)^{1/2}}$

$\Big\}$ depend on branch

- $\frac{d}{dz} \tanh^{-1} z = \frac{1}{1-z^2}$ doesn't depend on branch.

(Ex!)