Ambiguity of the notations: $\log z$ & $\operatorname{Log} z$

possible meanings

$\log z$	multiple-valued function (set) $\log z = \ln\lvert z\rvert + i \arg z$ $= \ln r + i(\theta + 2n\pi)$ $\theta \in \arg z, \quad n \in \mathbb{Z}$ $(r > 0)$	a branch of log (defined by some $\alpha \in \mathbb{R}$) $\log z = \ln\lvert z\rvert + i\theta$ $(\lvert z\rvert > 0) \quad \alpha < \theta < \alpha + 2\pi$ [and, for a fixed $z \in \mathbb{C} \setminus \{\text{ray in direction } \alpha\}$, $\log z$ is the value given by the formula]
$\operatorname{Log} z$	Principal Value $\operatorname{Log} z = \ln\lvert z\rvert + i\Theta$ $-\pi < \Theta \leq \pi$ $(\lvert z\rvert > 0)$	Principal Branch $(\alpha = -\pi)$ $\operatorname{Log} z = \ln\lvert z\rvert + i\Theta$ $(\lvert z\rvert > 0) \quad -\pi < \Theta < \pi$

only different in Principal Value and Principal Branch

(Sometimes we may write θ instead of Θ even in "principal" case)

Derivatives of $\log z$ (a branch of $\log z$)

Given a branch of $\log z$

$$\log z = \ln r + i\theta, \quad r>0, \ \underline{\alpha < \theta < \alpha + 2\pi}$$

$$(r = |z|)$$

We have

$$\begin{cases} \text{real part } & u = \ln r \\ \text{imaginary part } & v = \theta \end{cases}$$

$$\Rightarrow \begin{cases} u_r = \frac{1}{r} = \frac{1}{r} v_\theta \\ \frac{1}{r} u_\theta = 0 = -v_r \end{cases} \quad \text{CR-eqts (in polar form)}$$

$\therefore$ $u_r, u_\theta, v_r, v_\theta$ cts & satisfy CR-eqt. on

$$\{re^{i\theta} : r>0, \ \alpha < \theta < \alpha + 2\pi\}$$

$\Rightarrow$

(This branch of) $\log z$ is <u>analytic</u> on $r>0$, $\alpha < \theta < \alpha + 2\pi$.

and
$$\frac{d}{dz} \log z = e^{-i\theta}(u_r + i v_r)$$
$$= e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

In particular, if $\alpha = -\pi$, we have $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$.

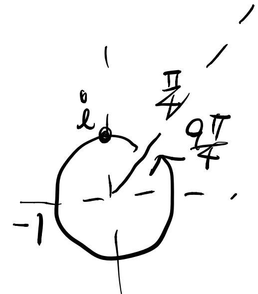
Conclusion :

$$\boxed{\begin{array}{l} \text{For any } \alpha, \quad \frac{d}{dz}\log z = \frac{1}{z} \quad \text{for } |z|>0,\ \alpha < \arg z < \alpha + 2\pi \\ \text{In particular}, \quad \frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z} \quad \text{for } |z|>0,\ -\pi < \mathrm{Arg}\, z < \pi. \end{array}}$$

Eg: $\log z^n \neq n \log z$ in general (Sometimes true, some times not)

(True) Take a branch of $\log z$

$$\log z = \ln r + i\theta, \quad \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$



Then $i^2 = -1 = e^{i\pi}$ in this branch

$\therefore \quad \log i^2 = i\pi \qquad \left(\frac{\pi}{4} < \pi < \frac{9\pi}{4}\right)$

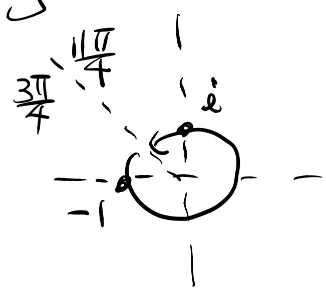
Also $i = e^{i\frac{\pi}{2}}$ in this branch

$\therefore \quad \log i = i\frac{\pi}{2}$

Hence in this branch $2\log i = \log i^2$

(Not true) Take a different branch

$$\log z = \ln r + i\theta, \quad \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$



Then $i^2 = -1 = e^{i\pi}$ in this branch $\left(\frac{3\pi}{4} < \pi < \frac{11\pi}{4}\right)$

but $i = e^{i\frac{5\pi}{2}}$ in this branch $\left(\frac{3\pi}{4} < \frac{5\pi}{2} < \frac{11\pi}{4}\right)$

Hence for this branch, $\begin{cases} \log i^2 = i\pi \\ \log i = i\frac{5\pi}{2} \end{cases}$

$$\Rightarrow 2\log i = i\,5\pi \neq i\pi = \log i^2$$

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§34 Some Identities involving Logarithmic

Prop: $\forall z_1, z_2 \in \mathbb{C}\setminus\{0\}$

$$\log(z_1 z_2) = \log z_1 + \log z_2$$ as multiple-valued functions (or sets of inverse images.)

(Pf: Omitted)

§35 The Power Function

<u>Def</u>: $\forall$ cpx number c, we define the power function by

$$z^c \underline{\underline{\text{def}}} e^{c \log z} \quad (\text{for } z \neq 0)$$

Notes: (1) z^c is possibly multiple-valued.

(2) For $c = n \in \mathbb{Z}$, then

$\Theta = \operatorname{Arg} z$, $k \in \mathbb{Z}$

$$z^n = e^{n \log z} = e^{n(\ln r + i(\Theta + 2k\pi))}$$

$$= e^{n \ln r + in\Theta + i\, 2nk\pi}$$

$$= r^n e^{in\Theta} e^{i\, 2nk\pi}$$

$$= r^n e^{in\Theta} = (re^{i\Theta})^n$$

$\therefore$ z^n is single-valued and equal to our original definition for z^n. ($z \neq 0$)

(3) From (2), for $n \geq 0$, we can see that $z^n = e^{n \log z}$ for $z \neq 0$ can be extended to a single-valued function z^n defined on the whole $\mathbb{C}$.

In particular, $\boxed{z^0 = 1, \ \forall z \in \mathbb{C}.}$

(4) For $c=\frac{1}{n}$, $n\geq 1$, $(z\neq 0)$

$$z^{\frac{1}{n}} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}[\ln r + i(\Theta + 2k\pi)]}, \quad k\in\mathbb{Z}$$

$$= e^{\frac{1}{n}\ln r + i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)}, \quad k\in\mathbb{Z}$$

$$= \sqrt[n]{r}\, e^{i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)}, \quad k=0,1,\cdots,n-1,$$

= set of n-roots of z.

<u>Def</u>: A <u>branch of</u> z^c is the function defined on the domain of a branch of $\log z$ with value given by the formula

$$z^c = e^{c\log z}, \quad r>0, \ \underline{\alpha<\theta<\alpha+2\pi},$$

with the corresponding branch of $\log z$.

<u>Prop</u>: For any <u>branch</u> of z^c

$$\frac{d}{dz} z^c = c z^{c-1}, \quad |z|>0, \ \underline{\alpha<\arg z<\alpha+2\pi}.$$

<u>Pf</u>: For the corresponding branch of $\log z$,

we have $\quad \frac{d}{dz}\log z = \frac{1}{z} \quad , |z|>0, \ \alpha < \arg z < \alpha + 2\pi.$

Hence $\quad \frac{d}{dz} z^c = \frac{d}{dz} e^{c\log z} = e^{c\log z} \frac{d}{dz}(c\log z)$

$$= c\, e^{c\log z} \cdot \frac{1}{z}$$

$$= c\, e^{c\log z} e^{-\log z}$$

$$= c\, e^{(c-1)\log z}$$

$$= c\, z^{c-1} . \quad \#$$