Pf: $|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2$

$$= \sin^2 x\,(1+\sinh^2 y) + \cos^2 x \sinh^2 y$$

$$= \sin^2 x + \sinh^2 y \qquad \left(\begin{array}{c}\text{by}\\ \cosh^2 y - \sinh^2 y = 1\\ \sin^2 x + \cos^2 x = 1\end{array}\right)$$

Similarly for

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad (\text{Ex!}) \quad \divideontimes$$

(Note: $|\sin z|$ and $|\cos z|$ are unbounded!)

§38 Zeros & Singularities of trigonometric functions

Def: A zero of a given function f is a cpx number z_0 such that $f(z_0) = 0$.

Thm The zeros of $\sin z$ and $\cos z$ in $\mathbb{C}$ are the same as the zeros of $\sin x$ and $\cos x$ on $\mathbb{R}$.

That is

$$\boxed{\begin{array}{ll}\sin z = 0 \iff z = n\pi & (n \in \mathbb{Z})\\ \cos z = 0 \iff z = \frac{\pi}{2} + n\pi & (n \in \mathbb{Z})\end{array}}$$

Pf: By property (7) of previous section

$$0 = |\sin z|^2 = \sin^2 x + \sinh^2 y$$

$\Leftrightarrow \quad \sin x = 0$ and $\sinh y = 0$

$$\sinh y = \frac{e^{y} - e^{-y}}{2}$$

$\Leftrightarrow \quad x = n\pi, \ n \in \mathbb{Z}$ and $y = 0$

$\Leftrightarrow \quad z = n\pi, \ n \in \mathbb{Z}.$

Similarly for $\cos z$. ※

<u>Def</u>: (Other trigonometric functions)

$$\tan z \stackrel{\text{def}}{=} \frac{\sin z}{\cos z}, \quad \cot z \stackrel{\text{def}}{=} \frac{\cos z}{\sin z}$$

$$\sec z \stackrel{\text{def}}{=} \frac{1}{\cos z}, \quad \csc z \stackrel{\text{def}}{=} \frac{1}{\sin z}.$$

<u>Note</u>: By the above thm,

(i) $\tan z$, $\sec z$ are analytic except at the singularities $z = \frac{\pi}{2} + n\pi, \ n \in \mathbb{Z}$.

(ii) $\cot z$, $\csc z$ are analytic except at the singularities $z = n\pi, \ n \in \mathbb{Z}$.

Prop:

$$\begin{cases} \frac{d}{dz}\tan z = \sec^2 z \\ \frac{d}{dz}\cot z = -\csc^2 z \\ \frac{d}{dz}\sec z = \sec z \tan z \\ \frac{d}{dz}\csc z = -\csc z \cot z \end{cases}$$

(Ex!)

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§39 Hyperbolic Functions

Recall for $y \in \mathbb{R}$,

$$\begin{cases} \sinh y = \dfrac{e^y - e^{-y}}{2} \\ \cosh y = \dfrac{e^y + e^{-y}}{2} \end{cases}$$

Hence we define

Def: $\forall z \in \mathbb{C}$,

$$\begin{cases} \sinh z = \dfrac{e^z - e^{-z}}{2} & (\text{No } i) \quad (\operatorname{sh} z) \\ \cosh z = \dfrac{e^z + e^{-z}}{2} & (\operatorname{ch} z) \end{cases}$$

the hyperbolic sine and hyperbolic cosine functions for cpx numbers respectively.

Properties

(1) $\begin{cases} \sinh(iz) = i\sin z, & \cosh(iz) = \cos z \\ \sin(iz) = i\sinh z, & \cos(iz) = \cosh z \end{cases}$

(2) $\begin{cases} \sinh(-z) = -\sinh z & \text{odd} \\ \cosh(-z) = \cosh z & \text{even} \end{cases}$

(3) $\cosh^2 z - \sinh^2 z = 1$

(4) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(5) $\begin{cases} \sinh z = \sinh x \cos y + i \cosh x \sin y \\ \cosh z = \cosh x \cos y + i \sinh x \sin y \end{cases}$

(6) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

$|\cosh z|^2 = \sinh^2 x + \cos^2 y$

(7) $\begin{cases} \sinh z = 0 \iff z = n\pi i, & n \in \mathbb{Z} \\ \cosh z = 0 \iff z = (\frac{\pi}{2} + n\pi) i, & n \in \mathbb{Z} \end{cases}$

(All proofs for (1)-(7) = Ex!)

(Note: by (6), $|\sinh z|$ & $|\cosh z|$ are <u>unbounded</u>!)

Ref: (Other hyperbolic functions)

$$\begin{cases} \tanh z \stackrel{\text{def}}{=} \dfrac{\sinh z}{\cosh z} & \text{(hyperbolic tangent)} \\ \coth z \stackrel{\text{def}}{=} \dfrac{\cosh z}{\sinh z} & \text{(" cotangent)} \\ \operatorname{sech} z \stackrel{\text{def}}{=} \dfrac{1}{\cosh z} & \text{(" ")} \\ \operatorname{csch} z \stackrel{\text{def}}{=} \dfrac{1}{\sinh z} & \text{(" ")} \end{cases}$$

Prop:

$$\begin{cases} \dfrac{d}{dz} \sinh z = \cosh z \\ \dfrac{d}{dz} \cosh z = \sinh z \end{cases}$$

$$\begin{cases} \dfrac{d}{dz} \tanh z = \operatorname{sech}^2 z \\ \dfrac{d}{dz} \coth z = -\operatorname{csch}^2 z \\ \dfrac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z \\ \dfrac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z \end{cases}$$

(Pf: Ex!)

Back to §31-36

§31 The Logarithmic function

Def: The (multiple-valued) logarithmic function of $z = re^{i\theta} \in \mathbb{C}\setminus\{0\}$ is

$$\log z \overset{\text{def}}{=\!=} \ln r + i \arg z$$

$$= \ln r + i(\theta + 2n\pi), \quad n \in \mathbb{Z}.$$

where $\ln r$ is the natural log for $r \in \mathbb{R}^+$

Note: (1) $\log z$ is a set and can be written as

$$\boxed{\log z = \ln|z| + i \arg z}$$

sets

(2) $\forall\, w \in \log z$, we have

$$e^w = e^{\ln|z| + i(\theta + 2n\pi)} \quad \text{for some } \theta \in \arg z$$

$$= e^{\ln|z|} e^{i\theta} e^{2n\pi i} = |z| e^{i\theta} = z$$

$\therefore$ $\log z$ is the "inverse" of $\exp z$!

(3) However

$$\log e^{z} = \log(e^{x+iy}) = \ln e^{x} + i(y+2n\pi), \quad n\in\mathbb{Z}$$

$$= x + iy + 2n\pi i$$

$$= z + 2n\pi i, \quad n\in\mathbb{Z}.$$

Def: Principal value of $\log z$ (for $z\neq 0$), denoted by $\text{Log}\, z$ is

$$\boxed{\text{Log}\, z = \ln|z| + i\,\text{Arg}\, z} \in \mathbb{C}$$

principal argument of z $\in(-\pi,\pi]$.

Note: Hence $\log z = \text{Log}\, z + 2n\pi i$, $n\in\mathbb{Z}$.

(§32 Omitted)

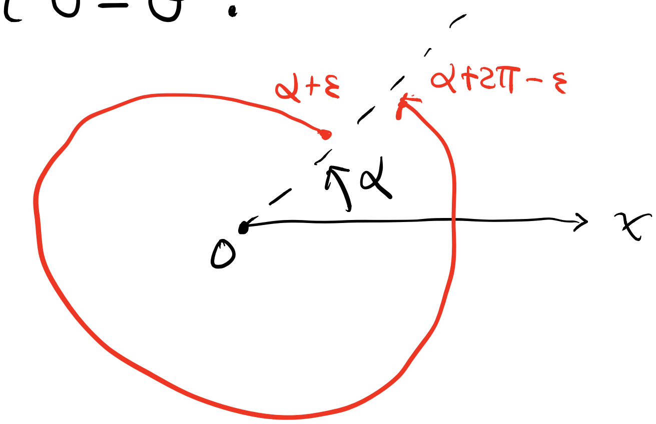
§33 Branches and Derivatives of Logarithmics

Def: For any $\alpha\in\mathbb{R}$, restriction of $\log z$ on the domain $\{re^{i\theta} : r>0,\ \alpha<\theta<\alpha+2\pi\}$ becomes a single-valued function

$(*)$ $\log z = \ln r + i\theta$, $r>0$, $\underline{\underline{\alpha<\theta<\alpha+2\pi}}$

with components $\begin{cases} u = \ln r \\ v = \theta \end{cases}$.

(Note: $\alpha \in \mathbb{R}$ is arbitrary, not necessary in $[-\pi, \pi]$)



Then single-valued function defined by $(*)$ is called a <u>branch of $\log z$</u>.

<u>Note</u>: For simplicity, we use the same notation $\log z$ for any branch and use the <u>restriction</u> on θ to distinguish the branches:

eg: • $\log z = \ln r + i\theta$, $r>0$, $\underline{\underline{\frac{3\pi}{4}<\theta<\frac{3\pi}{4}+2\pi}}$ $\left(= \frac{11\pi}{4}\right)$

• $\log z = \ln r + i\theta$, $r>0$, $\underline{\underline{-\frac{5\pi}{4}<\theta<\frac{3\pi}{4}}}$

are different branches of $\log z$

even the domains look the same

$$\{re^{i\theta} : r>0,\ \tfrac{3\pi}{4}<\theta<\tfrac{11\pi}{4}\}$$

$$=\{re^{i\theta} : r>0,\ -\tfrac{5\pi}{4}<\theta<\tfrac{3\pi}{4}\}$$

Caution: For a branch of $\log z$, then the notation $\log z$, for a fixed z, is the value of the function (branch of log) at the point z, not the set "$\log z$"!

So it is important to mention the branch.

Def: The branch defined by $\alpha = -\pi$

i.e. $r>0,\ \underline{\underline{-\pi<\theta<\pi}}$

is called the Principal Branch of log and denoted by

$$\boxed{\operatorname{Log} z = \ln r + i\Theta,\ r>0,\ \underline{\underline{-\pi<\Theta<\pi}}}$$

Notes: (1) The ray $\{\theta=\alpha\}$ is called the branch cut of the branch.

(2) The branch $\log z = \ln r + i\theta$, $r>0$, $\alpha<\theta<\alpha+2\pi$ cannot be extended continuously across the branch cut.

(3) Because of (2), we use open interval $\alpha<\theta<\alpha+2\pi$ for branch, but not semi-closed interval $\alpha<\theta\leq\alpha+2\pi$ as in the Principal value. $(-\pi<\Theta\leq\pi)$.