§27 Harmonic Functions

Def: A real-valued function $H = H(x,y)$ of 2-variables is said to be harmonic in a domain $D \subset \mathbb{R}^2$, if $H \in C^2(D)$ (has cts. 2^{nd} order partial derivatives) and satisfies $\boxed{H_{xx} + H_{yy} = 0}$ (Laplace equation)

Thm: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u, v are harmonic in D.

Pf: (Sketch :) f analytic $\Rightarrow$ CR-eqts $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

differentiate $u_x = v_y$ wrt x $\Rightarrow$ $u_{xx} = v_{yx}$

" $u_y = -v_x$ wrt y $\Rightarrow$ $u_{yy} = -v_{xy}$

$\Rightarrow$ $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ "since $v \in C^2(D)$".

Similarly, differentiate $u_x = v_y$ wrt y $\Rightarrow$ $u_{xy} = v_{yy}$

" $u_y = -v_x$ wrt x $\Rightarrow$ $u_{yx} = -v_{xx}$

$\Rightarrow$ $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$. #

eg (i) $f(z) = \sin x \cosh y + i \cos x \sinh y$ $\left(\begin{array}{l} \cosh y = \frac{e^y + e^{-y}}{2} \\ \sinh y = \frac{e^y - e^{-y}}{2} \end{array} \right)$

$= \quad u \quad + i \quad v$

$$\begin{cases} u_x = \cos x \cosh y & v_x = -\sin x \sinh y \\ u_y = \sin x \sinh y & v_y = \cos x \cosh y \end{cases}$$

$\therefore$ u_x, u_y, v_x, v_y all exist and cts on the whole $\mathbb{C}$

$\Rightarrow$ $f(z)$ is analytic on $\mathbb{C}$.

Verify: $u_{xx} + u_{yy} = (\cos x \cosh y)_x + (\sin x \sinh y)_y$

$= -\sin x \cosh y + \sin x \cosh y$

$= 0$

$\therefore$ u is harmonic.

Similarly for v.

eg (ii) Reading exercise : $f(z) = \frac{1}{z^2}$ analytic in $\mathbb{C} \setminus \{0\}$

$\Rightarrow$ $\frac{x^2 - y^2}{(x^2+y^2)^2}$ and $\frac{2xy}{(x^2+y^2)^2}$ harmonic on $\mathbb{C} \setminus \{0\}$.

(§28, 29 postponed.)

Ch 3 Elementary Functions

§30 The Exponential Function

Def: The exponential function e^z or $\exp z$ is defined by

$$\boxed{\exp z = e^z \overset{\text{def}}{=\!=} e^x(\cos y + i\sin y) \quad \text{for } z = x+iy \in \mathbb{C}}$$

(we usually write $e^z = e^x e^{iy}$,
where $e^{iy} = \cos y + i \sin y$)

Notation: "$\exp z$" is a better notation in the following situation:

for $z = \frac{1}{n}$, then $\exp(\frac{1}{n}) = e^{\frac{1}{n}} = \sum_{k=0}^{\infty} \frac{(\frac{1}{n})^k}{k!} \in \mathbb{R}$

which is the positive n-root of the real number $e = 2.71828\cdots$. ($\sqrt[n]{e}$)

This is in conflict with our convention that

$z_0^{\frac{1}{n}}$ = set of n-th roots of z_0 !

For convenient, we will accept this exception for $e^{\frac{1}{n}}$ and interpret it as the value $\exp(\frac{1}{n})$.

Properties :

(1) $|e^z| = e^x$, $\arg e^z = y + 2n\pi$, $n \in \mathbb{Z}$

(2) $e^z \neq 0$, $\forall z \in \mathbb{C}$.

(3) $\boxed{e^{z_1} e^{z_2} = e^{z_1+z_2}}$ (by compound angles formula)

$\Rightarrow \quad \dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

(4) $\boxed{\dfrac{d}{dz} e^z = e^z}$ $\Rightarrow$ e^z is entire

(see eg in §23)

(5) $e^{z+2\pi k i} = e^z$, $\forall k \in \mathbb{Z}$.

In particular $\boxed{e^{2\pi i} = 1}$

Lets study § 37 – 39 first.

§37 The Trigonometric functions $\sin z$ & $\cos z$

Euler formula $\Rightarrow$ $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$

$$e^{-ix} = \cos x - i \sin x$$

$$\Rightarrow \begin{cases} \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \\ \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} \end{cases}$$

Therefore, we define

Def: $\forall z \in \mathbb{C}$

$$\boxed{\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}}$$

Properties:

(1) $\sin z$, $\cos z$ are entire

$$\begin{cases} \dfrac{d}{dz} \sin z = \cos z \\ \dfrac{d}{dz} \cos z = -\sin z \end{cases} \quad (\text{Ex!})$$

(2) $\begin{cases} \sin(-z) = -\sin z & \text{odd} \\ \cos(-z) = \cos z & \text{even} \end{cases}$

(3) $e^{iz} = \cos z + i \sin z$ generalization of Euler formula to cpx numbers.

(4) $\begin{cases} \sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\ \cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \end{cases}$ (Ex!)

(5) $\sin^2 z + \cos^2 z = 1$ (Ex!)

(6) Real & Imaginary parts of $\sin z$ and $\cos z$

$$\begin{cases} \sin z = \sin x \cosh y + i \cos x \sinh y \\ \cos z = \cos x \cosh y - i \sin x \sinh y \end{cases}$$ (eg (1) in §27)

Pf: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{(-i)}{2} \cdot \left[e^{i(x+iy)} - e^{-i(x+iy)}\right]$

$$= \frac{-i}{2}\left[e^{-y+ix} - e^{y-ix}\right]$$

$$= \frac{-i}{2}\left[e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right]$$

$$= -\frac{i}{2}\left[-(e^{y} - e^{-y})\cos x + i(e^{y} + e^{-y})\sin x\right]$$

$$= \sin x \cosh y + i \cos x \sinh y .$$

Similarly for $\cos z$ (Ex!)

(7) $$\begin{cases} |\sin z|^2 = \sin^2 x + \sinh^2 y \\ |\cos z|^2 = \cos^2 x + \sinh^2 y \end{cases}$$

(next time)