## § Rules for (cpx) differentiation

If (cpx) derivatives of $f$ and $g$ exist at $z$, then

(1) $\frac{d}{dz} c = 0$, for const. $c$.

(2) $\forall$ integer $n \geq 1$, $\frac{d}{dz} z^n = n z^{n-1}$

(3) $\frac{d}{dz}(f \pm g) = \frac{df}{dz} \pm \frac{dg}{dz}$

(4) $\frac{d}{dz}(fg) = f(z)\frac{dg}{dz} + \frac{df}{dz} g(z)$

(5) If $g(z) \neq 0$, then $\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dz} - f\frac{dg}{dz}}{g^2}$.

<u>Chain Rule</u>: If $f$ has derivative at $z_0$, $g$ has derivatives at $f(z_0)$. Then $F(z) = g(f(z))$ has derivative at $z_0$ and

$$\boxed{F'(z_0) = g'(f(z_0))\, f'(z_0)}$$

i.e. $\frac{dF}{dz} = \frac{dg}{dw}\frac{df}{dz}$ where $w = f(z)$.

(All proofs are ex!)

## §21 Cauchy-Riemann Equations

Thm: Suppose that $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the partial derivatives $u_x, u_y, v_x, v_y$ exist at the point $(x_0, y_0)$ and satisfy the Cauchy-Riemann equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{at } (x_0, y_0)$$

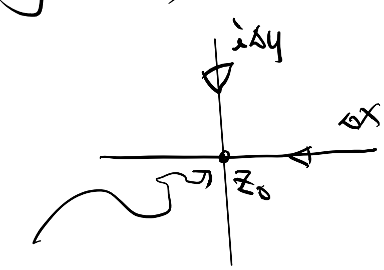
Also $f'(z_0) = u_x(x_0,y_0) + i v_x(x_0, y_0)$.

Pf: Let $z_0 = x_0 + iy_0$, $\Delta z = \Delta x + i\Delta y$

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

By assumption $\lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z} = f'(z_0)$ exists.

$\Rightarrow$ along any path of $\Delta z$ going to 0, we have the same limit $f'(z_0)$



In particular,

<u>Horizontal approach</u> $\Delta z = \Delta x \quad (\Delta y = 0)$

$$\Rightarrow \quad f'(z_0) = \lim_{\Delta x \to 0} \left[ \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \, \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \right]$$

$$= u_x(x_0,y_0) + i\, v_x(x_0,y_0) \quad \text{——} (*)$$

(i.e. $u_x, v_x$ exist at $(x_0,y_0)$ & $f' = u_x + i v_x$ at $(x_0,y_0)$)

<u>Vertical approach</u> $\Delta z = i\Delta y \quad (\Delta x = 0)$

$$\Rightarrow \quad f'(z_0) = \lim_{i\Delta y \to 0} \left[ \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \, \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y} \right]$$

$$= -i\, u_y(x_0,y_0) + v_y(x_0,y_0)$$

Comparing with $(*)$, we have

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{this is the CR-eqts.}$$

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§22 <u>Examples</u>

<u>eg1</u>: $f(z) = z^2$ is differentiable & $f'(z) = 2z$.

$$\begin{aligned} &\;\,\| \\ (x&+iy)^2 = (x^2-y^2) + 2ixy \end{aligned}$$

i.e. $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

$$\begin{cases} u_x = 2x \\ u_y = -2y \end{cases} \quad \& \quad \begin{cases} v_x = 2y \\ v_y = 2x \end{cases}$$

And satisfy $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ the CR-eqts.

(Interested students should see eg3. in this section of the textbook.)

## §23 Sufficient Condition for (cpx) Differentiability

**Thm** Let $f(z) = u(x,y) + i v(x,y)$ defined throughout some $\varepsilon$-nbd $B_\varepsilon(z_0)$ of $z_0 = x_0 + i y_0$, and

(a) $u_x, u_y, v_x, v_y$ exist everywhere in $B_\varepsilon(z_0)$

(b) $u_x, u_y, v_x, v_y$ are continuous at $(x_0, y_0)$ and satisfy $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at $(x_0, y_0)$

Then $f'(z_0)$ exists and $f'(z_0) = (u_x + i v_x)(x_0, y_0)$.

(Pf: Omitted)

eg: $f(z) = e^x \cos y + i e^x \sin y$ for $z = x + iy$.

Then $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$

$\Rightarrow \begin{cases} u_x = e^x \cos y \\ u_y = -e^x \sin y \end{cases}$ & $\begin{cases} v_x = e^x \sin y \\ v_y = e^x \cos y \end{cases}$

$u_x, u_y, v_x, v_y$ exist and are continuous functions on the whole $(x,y)$ plane; satisfies

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{CR eqts.}$$

(by Thm)

$\Rightarrow$ $f(z) = e^x \cos y + i e^x \sin y$ is (complex) differentiable at any $z \in \mathbb{C}$,

and $f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$

$= f(z)$.

$\left( f(z) = e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z \right)$

## §24 Polar coordinates

**Thm**: Let $f(z) = u(r,\theta) + i v(r,\theta)$ be defined in some $\varepsilon$-nbd of a non-zero point $z_0 = r_0 e^{i\theta_0}$, and suppose that

(a) $u_r, u_\theta, v_r, v_\theta$ exist everywhere in the $\varepsilon$-nbd,

(b) $u_r, u_\theta, v_r, v_\theta$ continuous at $(r_0, \theta_0)$ satisfying

$$\begin{cases} u_r = \frac{1}{r} v_\theta \\ \frac{1}{r} u_\theta = - v_r \end{cases}$$

(Polar form of CR-eqts) at $(r_0, \theta_0)$.

Then $f'(z_0)$ exists, and

$$f'(z_0) = e^{-i\theta_0} \left( u_r(r_0,\theta_0) + i\, v_r(r_0,\theta_0) \right).$$

(Pf: Change of variables (Ex!))

**eg**: If $f(z) = \frac{1}{z^2} = \frac{1}{r^2 e^{i2\theta}} = \frac{1}{r^2} e^{-i2\theta}$ (for $z = re^{i\theta} \neq 0$)

$$= \frac{1}{r^2} \cos 2\theta - i \frac{1}{r^2} \sin 2\theta$$

i.e. $\begin{cases} u = \frac{1}{r^2}\cos 2\theta \\ v = -\frac{1}{r^2}\sin 2\theta \end{cases}$

$$\begin{cases} u_r = -\frac{2}{r^3}\cos 2\theta \\ \frac{1}{r}u_\theta = -\frac{2\sin 2\theta}{r^3} \end{cases} \quad \& \quad \begin{matrix} v_r = \frac{2}{r^3}\sin 2\theta \\ \frac{1}{r}v_\theta = -\frac{2}{r^3}\cos 2\theta \end{matrix}$$

$\therefore$ $u_r, u_\theta, v_r, v_\theta$ exist & cts at any $r, \theta$ $(r \neq 0)$

$\Rightarrow$ $f(z) = \frac{1}{z^2}$ is cpx differentiable everywhere in $\mathbb{C}\setminus\{0\}$

& $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(-\frac{2}{r^3}\cos 2\theta + i\frac{2}{r^3}\sin 2\theta\right)$$

$$= -\frac{2}{r^3}e^{-i\theta}(\cos 2\theta - i\sin 2\theta)$$

$$= -\frac{2}{r^3}e^{-i\theta}e^{-i2\theta} = -\frac{2}{r^3 e^{i3\theta}}$$

$$= -\frac{2}{z^3}$$

i.e. $\left(\frac{1}{z^2}\right)' = -\frac{2}{z^3}$ #

## § 25 Analytic Functions

Def: (1) A function $f(z)$ is analytic at a point $z_0$ if $f'(z)$ exists $\forall z \in B_\varepsilon(z_0)$ for some $\varepsilon > 0$.

(2) A function $f$ said to be analytic in a set $S$, if $f$ is analytic at all $z \in S$.

(3) An entire function is a function analytic on the whole complex plane.

egs (i) $\frac{1}{z}$ is analytic in $0 < |z| < \infty$ (Ex!)

(ii) $f(z) = |z|^2$ is differentiable at $z=0$, but not analytic at $z=0$ (since $f$ is not differentiable for $z \neq 0$)

(iii) Polynomials $a_0 + a_1 z + \cdots + a_n z^n$ are entire.

(iv) $f(z) = e^x \cos y + i e^x \sin y$ is entire.
$(= e^z)$

Simply properties

(i) $f$ analytic in $D \Rightarrow f$ continuous in $D$

(ii) Analytic in $D \Rightarrow$ CR-eqts. in $D$

(iii) $\forall$ 1st order partial derivatives exist & cts on $D$
+ CR-eqts. everywhere in $D$ (provided $\forall z \in D$ $\exists B_\varepsilon(z) \subset D$)
$\Rightarrow$ analytic in $D$.

(iv) $f, g$ analytic $\Rightarrow$ $\begin{cases} f \pm g,\ fg \text{ analytic} \\ \frac{f}{g} \text{ analytic provided } g \neq 0 \end{cases}$

(In particular, rational function $\frac{P(z)}{Q(z)}$ is analytic in $\{z : Q(z) \neq 0\}$.)

(v) $f, g$ analytic $\Rightarrow$ $f \circ g$ analytic &
$(f \circ g)' = f'(g) g'$.

Thm: Let $D$ be a domain (open and connected)
($\forall z \in D, \exists \varepsilon > 0$, s.t. $B_\varepsilon(z) \subset D$)
If $f'(z) = 0$ $\forall z \in D$,

then $f(z) = \text{constant}$, $\forall z \in D$.

<u>Pf</u>: Let $f(z) = u + iv$.

Then $\quad 0 = f'(z) = u_x + i v_x$

$$\Rightarrow \quad u_x = v_x = 0 \quad \forall z \in D$$

CR-eqts $\Rightarrow \quad v_y = -u_y = 0 \quad \forall z \in D$

Since $D$ is connected, by a Thm in Advanced Calculus,

$$\left\{ \begin{array}{l} u = u_0 \\ v = v_0 \end{array} \right. \text{(constants)}, \ \forall z \in D$$

$$\therefore \quad f = u_0 + i v_0 \quad \text{(constant)}, \ \forall z \in D.$$

\#

<u>Def</u>: A point $z_0$ is called a <u>singular point</u> of $f$ if $f$ is <u>not</u> analytic at $z_0$ but $\exists$ seq. $z_n \to z_0$ s.t. $f$ is analytic at $z_n$, $\forall n$.

<u>eg</u>: $z = 0$ is a singular point of $f(z) = \frac{1}{z}$.

($f$ not even defined at $z=0$, but analytic $\forall z \neq 0$)

## §26 <u>Further Examples</u>

eg1: $f(z) = \dfrac{z^2+3}{(z+1)(z^2+5)}$ is analytic in $\mathbb{C}\setminus\{-1, \pm i\sqrt{5}\}$

$\Rightarrow$ $-1, \pm i\sqrt{5}$ are singular points of $f$.

eg3 (Prop): If $f = u+iv$, $\bar{f} = u - iv$ are both analytic in a domain $D$ (open & connected). Then $f$ = constant on $D$.

Pf: $\bar{f}$ analytic $\Rightarrow$ $\begin{cases} u_x = (-v)_y = -v_y \\ u_y = -(-v)_x = v_x \end{cases}$

Together with $f$ analytic $\Rightarrow$ $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

we have $\begin{cases} u_x = -u_x \\ u_y = -u_y \end{cases}$ $\Rightarrow$ $u_x = u_y = 0$

$\Rightarrow$ $v_x = v_y = 0$ also.

Hence "$D$ domain" $\Rightarrow$ $u$ = const., $v$ = const.

& hence $f = u + iv$ = const. $\times\!\!\times$

eg4 (Prop) If $f$ is analytic on a domain $D$,
and $|f| \equiv$ const. on $D$,
then $f$ = const. on $D$.

Pf: Let $|f| \equiv r_0$ a real constant on $D$.
If $r_0 = 0$, then $f \equiv 0$ on $D$. We're done.

Assume $r_0 \neq 0$, then $f(z) \neq 0$, $\forall z \in D$.

$$\Rightarrow \quad \overline{f}(z) = \frac{|f|^2}{f(z)} = \frac{r_0^2}{f(z)} \text{ analytic on } D.$$

By eg3, $f \equiv$ const. on $D$. ※