Ch1 Complex Numbers

Standard notations

$$\begin{cases} \mathbb{N} = \{0,1,2,3,\cdots\} & \text{set of natural numbers} \\ \mathbb{Z} = \{\cdots,-3,-2,-1,0,1,2,3,\cdots\} & \text{set of integers} \\ \mathbb{Q} = \text{set of rational numbers} \\ \mathbb{R} = \text{set of real numbers} \end{cases}$$

§1 Sums & Product

Def: The set of complex numbers $\mathbb{C}$ is

$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ with pair (x,y) denoted

by $\underline{z = x+iy}$ endowed with the following operations

$$\begin{array}{ll} \text{(sum)} & (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2) \\ \text{(product)} & (x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2) \end{array}$$

eg: $i^2+1=0$ (Check!)

Notes: (1) If $z = x+iy$, then $x = \operatorname{Re} z$ (real part)
$y = \operatorname{Im} z$ (imaginary part)

(2) "sum" is also referred as "addition";
"product" " " " " "multiplication".

§2&3 Basic Algebraic Properties

$$\left\{\begin{array}{l} z_1+z_2=z_2+z_1 \\ (z_1+z_2)+z_3=z_1+(z_2+z_3) \\ \exists\, 0 \text{ s.t. } z+0=z,\ \forall z \\ \forall z,\ \exists\, -z \text{ s.t. } z+(-z)=0 \end{array}\right.$$

$$\left\{\begin{array}{l} z_1z_2=z_2z_1 \\ (z_1z_2)z_3=z_1(z_2z_3) \\ \exists\, 1 \text{ s.t. } z1=z \\ \forall z\neq 0,\ \exists\, z^{-1} \text{ s.t. } zz^{-1}=1 \end{array}\right.$$

- $z_1(z_2+z_3)=z_1z_2+z_1z_3$

Notes: (1) $\forall n\in\mathbb{N}$, z^n is defined by induction

$$\left\{\begin{array}{l} z^{n+1}=z^nz \quad \text{with} \\ z^0=1 \end{array}\right.$$

(2) $z_1z_2=0 \Rightarrow z_1=0$ or $z_2=0$

(3) <u>subtraction</u> $\quad z_1 - z_2 \overset{\text{def}}{=\!=} z_1 + (-z_2)$

<u>division</u> $\quad \dfrac{z_1}{z_2} \overset{\text{def}}{=\!=} z_1 z_2^{-1} \quad (\text{for } z_2 \neq 0)$

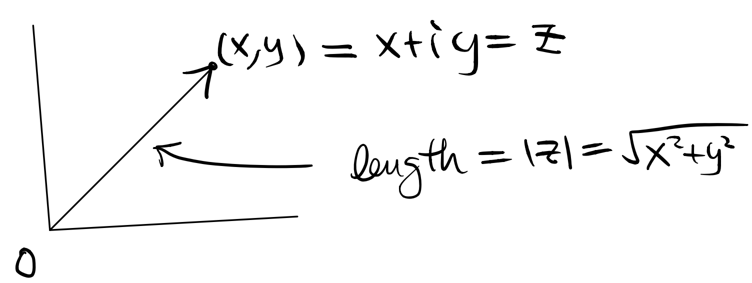
(4) <u>Binomial formula</u> is also hold for cpx numbers

$$(z_1+z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k},$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, $(k=0,1,2,\cdots)$ (Ex!)

§4 & 5 <u>Vectors & Moduli</u>, <u>Triangle inequality</u>

By definition of cpx number, $z = x+iy$ is naturally identified as a <u>plane vector</u> (x,y) in $\mathbb{R}^2$



<u>Note</u>: cpx number addition coincides with the vector addition.

Def: The <u>modulus</u>, or <u>absolute value</u>, of $z = x+iy$ is defined by $|z| = \sqrt{x^2+y^2}$

i.e. $|z|$ = length of the vector (x,y)
= distance between the points (x,y) & $(0,0)$.

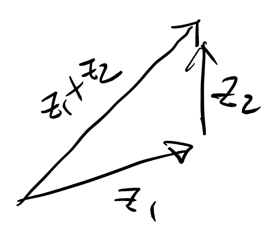
Notes: (1) The inequality $z_1 < z_2$ is <u>not</u> defined for cpx numbers. Therefore $z_1 < z_2$ is <u>meaningless</u> unless $z_1, z_2 \in \mathbb{R}$.

However, <u>$|z_1| < |z_2|$ is meaningful</u>!

(2) Easy to prove that

$$\begin{cases} \operatorname{Re} z \le |\operatorname{Re} z| \le |z| \\ \operatorname{Im} z \le |\operatorname{Im} z| \le |z| \end{cases} \qquad (\text{Ex!})$$

(3) <u>Triangle inequality</u>



$$|z_1 + z_2| \le |z_1| + |z_2|$$

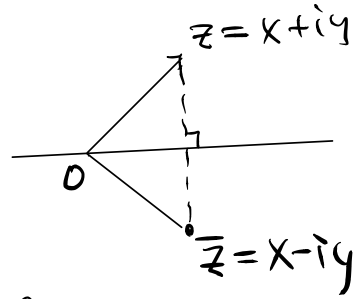
and hence

$$\big|\,|z_1| - |z_2|\,\big| \le |z_1 \pm z_2| \le |z_1| + |z_2| \qquad (\text{Ex!})$$

§6 Complex Conjugate

Def: The complex conjugate (or simply conjugate) of $z = x+iy$ is

$$\boxed{\bar{z} = x - iy}$$



i.e. $\bar{z}$ is represented by the reflection in real axis

It is easy to prove (Ex!)

- $$\begin{cases} \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \\ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2} \end{cases}$$

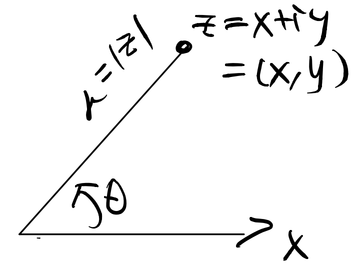
- $$\begin{cases} x = \mathrm{Re}\, z = \dfrac{z+\bar{z}}{2} \\ y = \mathrm{Im}\, z = \dfrac{z-\bar{z}}{2} \end{cases}$$

- $z\bar{z} = |z|^2 \quad \left((x+iy)(x-iy) = x^2+y^2\right)$

§7 Exponential Form

Using polar coordinate (r, θ) for $(x, y) = z = x + iy$,

we can write



$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and for some $\theta \in \mathbb{R}$.

Notes: (1) θ is undefined for $z = 0$

(2) θ is only defined up to $2k\pi$, $k \in \mathbb{Z}$

i.e. if θ satisfies $z = |z|(\cos\theta + i\sin\theta)$,

then $\forall\, k \in \mathbb{Z}$, we also have

$$z = |z|\left(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right)$$

Definitions

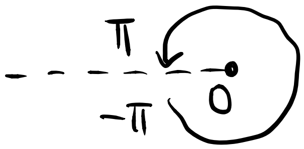
(1) Each value of θ s.t. $z = |z|(\cos\theta + i\sin\theta)$

is called an argument of z

(2) $\boxed{\arg z = \text{set of all arguments of } z}$

(3) The principal value of $\arg z$, or principal

__argument__ of z, denoted by $\underline{\operatorname{Arg} z}$ is the value $\Theta \in \arg z$ s.t. $-\pi < \Theta \le \pi$



(Arg z is discontinuous along negative real axis)

$\Rightarrow$ $\arg z = \{ \operatorname{Arg} z + 2k\pi : k \in \mathbb{Z} \}$ (is a set)

$\overset{\text{(write)}}{=} \operatorname{Arg} z + 2k\pi, \quad k \in \mathbb{Z}$ (for simplicity)

with $\operatorname{Arg} z \in (-\pi, \pi]$

Egs (1) $z = -1$, then $\operatorname{Arg}(-1) = \pi$ (not $-\pi$)

$\Rightarrow \arg(-1) = \{ \dots, \pi - 4\pi, \pi - 2\pi, \pi, \pi + 2\pi, \pi + 4\pi, \dots \}$

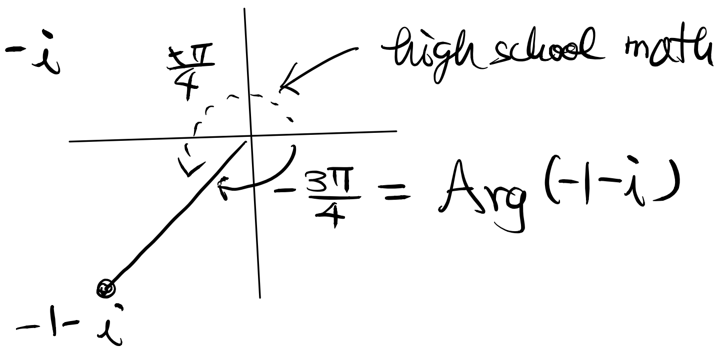
$= \pi + 2k\pi, \quad k \in \mathbb{Z}$ Arg(−1)

$= \{ \dots, -3\pi, -\pi, \pi, 3\pi, 5\pi, \dots \}$

$= -\pi + 2l\pi, \quad l \in \mathbb{Z}$ $(l = k+1)$

this is not __principal__.

(2) $z = -1 - i$



$-\frac{3\pi}{4} = \mathrm{Arg}(-1-i)$

$$\arg(-1-i) = -\frac{3\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$$
$$= \left\{ \cdots, -\frac{3\pi}{4}, \frac{5\pi}{4}, \cdots \right\}$$

<u>Notation</u>:

Define $\underline{e^{i\theta} \overset{\text{def}}{=} \cos\theta + i\sin\theta \ , \ \forall \theta \in \mathbb{R}}$ (Euler formula)

Then $z = r(\cos\theta + i\sin\theta) = \underline{\underline{|z| e^{i\theta}}}$

is called the <u>exponential form</u> of z.

i.e. $\boxed{z = |z| e^{i \arg z}} = |z| \exp[i \arg z]$

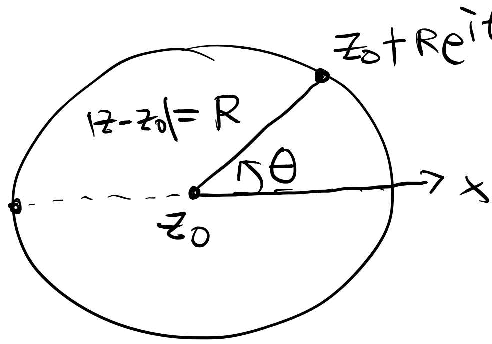
<u>eg3</u>: $z = -1-i$, $\mathrm{Arg}(-1-i) = -\frac{3\pi}{4}$

$$\therefore \ -1-i = |z| e^{-i\frac{3\pi}{4}}$$
$$= \sqrt{2}\, e^{-i\frac{3\pi}{4}} = \sqrt{2} \exp\left[-i\frac{3\pi}{4}\right]$$

$$(\text{of course}) = \sqrt{2}\exp\left(i\frac{5\pi}{4}\right)$$
$$= \sqrt{2}e^{i\frac{5\pi}{4}}$$

in fact
$$-1-i = \sqrt{2}e^{i\left(-\frac{3\pi}{4}+2k\pi\right)}$$
$$= \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}+2k\pi\right)\right], \quad k\in\mathbb{Z}.$$

Note: We can represent a circle of radius R centered at z_0 by
$$z = z_0 + Re^{i\theta}, \quad \theta\in(-\pi,\pi]$$



§8 Products & Powers in Exponential Form

Fact:
$$\boxed{e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}}$$

(Pf: Ex by compound angle formula)

Then, if $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$ $(z_1, z_2 \neq 0)$

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

  $\Rightarrow \quad |z_1 z_2| = |z_1||z_2|$

- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

  $\Rightarrow \quad z^{-1} = \dfrac{1}{z} = \dfrac{1}{r} e^{-i\theta}$ (for $z = re^{i\theta}$)

- $z^n = r^n e^{in\theta}$ (for $z = re^{i\theta}$)

$\Rightarrow$

<u>deMeivre's formula</u>

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

<u>Application</u> : $n=2 \Rightarrow \begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{cases}$

$n=3 \Rightarrow \begin{cases} \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \end{cases}$ (Ex!)