THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Extra Tutorial 2 (November 9)

Question 1. *Is there a function* $f : \mathbb{R} \to \mathbb{R}$ *that is continuous on* $\mathbb{R} \setminus \mathbb{Q}$ *but discontinuous on* $\mathbb{Q}$*?*

Answer: Yes. The Thomae's function.

Question 2. *Is there a function* $f : \mathbb{R} \to \mathbb{R}$ *that is continuous on* $\mathbb{Q}$ *but discontinuous on* $\mathbb{R} \setminus \mathbb{Q}$*?*

Answer: No. Follow the exercises below to give a proof of this result.

Exercise 1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function that is continuous on* $\mathbb{Q}$*. Write* $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$*.*

(a) Show that there is a sequence $\{I_n\}_{n\in\mathbb{N}}$ *of closed bounded intervals such that for all* $n \in \mathbb{N}$*,*

- *(i)* $\mathring{I}_n \supseteq I_{n+1}$ *(here* $\mathring{[a, b]} = (a, b)$*);*
- *(ii)* $I_n \cap \{r_1, \ldots, r_n\} = \emptyset$*;*
- *(iii)* $x, y \in I_n \implies |f(x) - f(y)| < \dfrac{1}{2^n}$.

(b) Show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ *and* f *is continuous on* $\bigcap_{n=1}^{\infty} I_n$*.*

(c) Hence conclude f *cannot be discontinuous on* $\mathbb{R} \setminus \mathbb{Q}$*.*

The following could be useful in proving (a) above.

Exercise 2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and let* $a < x_0 < b$*. Suppose* f *is continuous at* x_0*. Show that, given any* $\varepsilon > 0$*, there exist* $c, d \in \mathbb{R}$ *such that* $a < c < x_0 < d < b$ *and*

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever } x, y \in [c, d].$$