THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 9 (November 7)

The following were discussed in the tutorial this week:

1. Give an example for each of the following:
   (a) $f : \mathbb{R} \to \mathbb{R}$ continuous only at one point,
   (b) $f : \mathbb{R} \to \mathbb{R}$ discontinuous everywhere but $|f|$ continuous everywhere,
   (c) $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R} \backslash \mathbb{Q}$ but distcontinuous on $\mathbb{Q}$,

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x \to -\infty} f = 0$ and $\lim_{x \to \infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

3. **(Alternative proof of Location of Roots Theorem)** Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$. Prove that $f(w) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that for any finite collection $\{x_1, x_2, \ldots, x_n\}$ of real numbers, there exists $x \in \mathbb{R}$ such that

$$f(x) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$