THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 7 (October 24)

The following were discussed in the tutorial this week:

1. Use ε-δ definition to evaluate the $\lim_{x\to 2} \dfrac{x^3+1}{(x-1)(x-3)}$.

2. Recall the **Sequential Criteria for limit** and **Divergence Criteria for limit**.

3. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Suppose c is a cluster point of A.

   (a) Suppose $\lim_{x\to c} f(x)$ does not exist.
   Show that there exists $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in $A\backslash\{c\}$, both converging to c, such that $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$.

   (b) Prove the **Cauchy Criterion for limit**:
   $\lim_{x\to c} f(x)$ exists if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in A$ with $0 < |x-c|, |y-c| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   (a) Show that f has a limit at $x = 0$.

   (b) Show that if $c \neq 0$, then f does not have a limit at c.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0, l \in \mathbb{R}$. Suppose

   (i) $\lim_{x\to x_0} g(x) = y_0$ and $\lim_{y\to y_0} f(y) = \ell$;

   (ii) there exists $\delta > 0$ such that $g(x) \neq y_0$ whenever $0 < |x - x_0| < \delta$,

   (a) Show that $\lim_{x\to x_0} f(g(x)) = \ell$.

   (b) Can we drop condition (ii)?