TA's selected solution to 2050B optional test

3. (a). We claim that the assertion is false. To show that h is not uniformly continuous on $(0, \infty)$, by definition we need to find an $\varepsilon_0 > 0$ such that not matter which $\delta > 0$ is given, there exist some x and u in $(0, \infty)$ with $|x - u| < \delta$ such that $|h(x) - h(u)| \geq \varepsilon_0$.

Observe that for $0 < \delta \leq 1$, if $x := \delta$ and $u := \delta/2$, then $|x - u| = \delta/2 < \delta$ and $|h(x) - h(u)| = 1/\delta \geq 1$.

Hence, we take $\varepsilon_0 = 1$. Now given any $\delta > 0$, we have $x := \min(\delta, 1)$ and $u := \min(\delta, 1)/2$ satisfying $x, u \in (0, \infty)$, $|x - u| < \delta$ and $|h(x) - h(u)| \geq 1$, which was to be demonstrated.

(b). We claim that the assertion is true. To show that g is uniformly continuous on $[3, \infty)$, by definition we need to show that given any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if x and u are in $[3, \infty)$ with $|x - u| < \delta_\varepsilon$, then $|g(x) - g(u)| < \varepsilon$.

Observe that for $x, u \in [3, \infty)$,

$$\begin{aligned} |g(x) - g(u)| &= \frac{|x - u|}{|xu|} \\ &\leq \frac{|x - u|}{3 \cdot 3} \\ &\leq |x - u| \, . \end{aligned}$$

Hence, no matter which $\varepsilon > 0$ is given, if we take $\delta_\varepsilon := \varepsilon$, then for all $x, u \in [3, \infty)$ with $|x - u| < \delta_\varepsilon$, we have $|g(x) - g(u)| \leq |x - u| < \delta_\varepsilon = \varepsilon$. This proves our claim.

(c). We claim that the assertion is true.

First approach (proof by contradiction):*

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*If you have tried to think of the graph of f, finding that very likely the assertion is true, but are unable to give a direct proof, then proof by contradiction may help.

Suppose f does not attain both its absolute maximum and absolute minimum. Fix $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Since $\lim_{x\to\infty} f(x) = \ell = \lim_{x\to-\infty} f(x)$, there exists $M_\varepsilon > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in (-\infty, -M_\varepsilon) \cup (M_\varepsilon, \infty)$. This implies

$$\ell - \varepsilon < f(x) < \ell + \varepsilon$$

for all $x \in (-\infty, -M_\varepsilon) \cup (M_\varepsilon, \infty)$.

Now applying Maximum-Minimum Theorem to f on $[-M_\varepsilon, M_\varepsilon]$, we see that $\exists x^*, x_* \in [-M_\varepsilon, M_\varepsilon]$ such that

$$f(x_*) \leq f(x) \leq f(x^*)$$

for all $x \in [-M_\varepsilon, M_\varepsilon]$.

If it happened that $f(x^*) \geq \ell + \varepsilon$, then $f(x^*) \geq f(x)$ for all $x \in \mathbb{R}$, regardless of whether $x \in [-M_\varepsilon, M_\varepsilon]$ or not. It follows that f attains its absolute maximum at x^*, which contradicts our initial assumption. Therefore it must be that $f(x^*) < \ell + \varepsilon$, and so $f(x) \leq f(x^*) < \ell + \varepsilon$ for all $x \in [-M_\varepsilon, M_\varepsilon]$. By essentially the same argument, we have $\ell - \varepsilon < f(x_*) \leq f(x)$ for all $x \in [-M_\varepsilon, M_\varepsilon]$.

Combining the results, we have

$$\ell - \varepsilon < f(x) < \ell + \varepsilon$$

for all $x \in (-\infty, -M_\varepsilon) \cup (M_\varepsilon, \infty) \cup [-M_\varepsilon, M_\varepsilon] = \mathbb{R}$.

In particular, for our initially fixed $x_0 \in \mathbb{R}$, we have $\ell - \varepsilon < f(x_0) < \ell + \varepsilon$ and so $|f(x_0) - \ell| < \varepsilon$. The choice of $\varepsilon > 0$ is arbitrary, whence $f(x_0) = \ell$. Also, $x_0 \in \mathbb{R}$ is arbitrary, so it must be that $f(x) \equiv \ell$ on $\mathbb{R}$. But then f is a constant function which must attains both its absolute maximum and absolute minimum, contradicting our initial hypothesis. Done.

Second approach (direct proof):

If $f(x) \equiv \ell$ on $\mathbb{R}$, then plainly the assertion is true. Therefore, let's assume $\exists x_0 \in \mathbb{R}$ such that $f(x_0) \neq \ell$. We first assume $f(x_0) > \ell$; the case $f(x_0) < \ell$ will be handled later.

Since $f(x_0) > \ell$, $\varepsilon_0 := f(x_0) - \ell$ is a positive number. Since $\lim_{x\to\infty} f(x) = \ell = \lim_{x\to-\infty} f(x)$, there exists $M > 0$ such that $|f(x) - \ell| < \varepsilon_0$ for all $x \in (-\infty, -M) \cup (M, \infty)$. This implies

$$f(x) - \ell \leq |f(x) - \ell| < f(x_0) - \ell$$

for all $x \in (-\infty, -M) \cup (M, \infty)$, and so

$$f(x) < f(x_0)$$

for all $x \in (-\infty, -M) \cup (M, \infty)$.

Now applying Maximum-Minimum Theorem to f on $[-M, M]$, we see that $\exists x^* \in [-M, M]$ such that

$$f(x) \leq f(x^*)$$

for all $x \in [-M, M]$. Since $f(x) < f(x_0)$ on $(-\infty, -M) \cup (M, \infty)$, so $x_0 \notin (-\infty, -M) \cup (M, \infty)$ (otherwise $f(x_0) < f(x_0)$). Hence, $x_0 \in [-M, M]$ and so $f(x_0) \leq f(x^*)$.

It follows that $f(x) \leq f(x^*)$ for all $x \in \mathbb{R}$: if $x \in (-\infty, -M) \cup (M, \infty)$ then $f(x) < f(x_0) \leq f(x^*)$; else if $x \in [-M, M]$, then $f(x)$ is not greater than the absolute maximum of f on $[-M, M]$, which is $f(x^*)$. This means f attains its absolute maximum at x^*.

Finally, the case $f(x_0) < \ell$ can be handled in essentially the same way. This time we start with $f(x_0) < f(x)$ on some $(-\infty, -M) \cup (M, \infty)$ and argue that f attains its absolute minimum.