Solution to 2050B assignment 3

1. Please refer to textbook 3.2.11 Theorem.

2. (2 marks)

   Let $\beta := (1+x)/2$. We have $x < \beta < 1$. By homework II 4(b), $\exists N_1 \in \mathbb{N}$ such that $x_n^{1/n} < \beta$ for all $n > N_1$. Therefore, for all $n > N_1$, we have $0 \leq x_n \leq \beta^n$.

   Since $\lim \beta^n = 0$ (by e.g. question 1, ratio-test), so by squeeze theorem it follows that $\lim x_n = 0$.

3.
   - Simply,
   $$\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0.$$
   - We shall show the following statement: $\forall \varepsilon_0 > 0$,
   $$\lim_{n\to\infty} \frac{n^2}{(1+\varepsilon_0)^n} = 0.$$
   In particular we can take $\varepsilon_0 = 1$. The statement follows from question 1, ratio-test. Note that the terms are all positive, and that
   $$\lim_{n\to\infty} \left[\frac{(n+1)^2}{(1+\varepsilon_0)^{n+1}} \div \frac{n^2}{(1+\varepsilon_0)^n} \right] = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{(1+\varepsilon_0)} = \frac{1}{(1+\varepsilon_0)} < 1.$$
   (the following proposition is used for the last equality: for convergent sequences $(a_n), (b_n)$, we have $\lim(a_n + b_n) = \lim a_n + \lim b_n$ and $\lim a_n b_n = \lim a_n \cdot \lim b_n$)
   - Observe that
   $$\lim_{n\to\infty} \frac{2^n}{100^n} = \lim_{n\to\infty} \frac{1}{50^n} = 0.$$
   (the last equality follows from e.g. question 2, root-test)
   - We shall use question 1, ratio test to show that
   $$\lim_{n\to\infty} \frac{100^n}{n!} = 0.$$

Note that the terms are all positive, and that

$$\lim_{n\to\infty}\left[\frac{100^{n+1}}{(n+1)!}\div\frac{100^n}{n!}\right]=\lim_{n\to\infty}\frac{100}{n+1}=0<1.$$

(the last equality follows from e.g. Archimedean property)

- Observe that

$$\begin{aligned}\frac{n!}{n^n}&=\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{2}{n}\cdot\frac{1}{n}\\&\le 1\cdot 1\cdots 1\cdot\frac{1}{n}\\&=\frac{1}{n}.\end{aligned}$$

Therefore, by squeeze theorem, we see that $(\frac{n!}{n^n})$ converges to zero.

4. (2 marks)

   (a). Let $\delta_n := n^{1/n}-1$. We have $\delta_n \ge 1^{1/n}-1=0$, so by Binomial Theorem, we have $n=(1+\delta_n)^n \ge 1+\frac{n(n-1)}{2}\delta_n^2$, whence $\delta_n \le \sqrt{\frac{2}{n}}$ for all $n\ge 1$. Therefore, for all $n \ge 1$, we have

   $$1 \le n^{1/n} \le 1+\sqrt{\frac{2}{n}}$$

   Since $\lim(1+\sqrt{2/n})=1$, so by squeeze theorem we conclude that $\lim n^{1/n}=1$.

   (b). $\forall n\in\mathbb{N}$, we have $n\le n^2$, so $\frac{1}{n^2}\le\frac{1}{n}$ and $1\le n^{1/n^2}\le n^{1/n}$. By squeeze theorem and part (a), we have $\lim n^{1/n^2}=1$.

5. (2 marks) Since $s_{n+1}-s_n=x_{n+1}>0$, we see that (s_n) is an increasing sequence which is bounded above by M. By monotone convergence theorem, $\lim s_n$ exists.

6. (2 marks) Please refer to textbook 3.7.7 Comparison Test.

7. (2 marks) Please refer to textbook 9.1.2 Theorem.