TAs' solution[1] to 2050B assignment 2

1. (3 marks)

   - $\min A$ does not exist.
     If $m_0 = \min A$ exists, then by the definition of "min", we have $m_0 \in A$ and $m_0 \leq a$ $\forall a \in A$. Therefore $0 < m_0 \leq a \leq 1$ for all $a \in A$ (the first and last inequality come from the definition of A).
     Note that $m_0/2 \in A$ as $0 < m_0 \leq 1$. Thus we have $0 < m_0 \leq m_0/2$, which is impossible.
     Hence $\min A$ does not exist.

   - $\inf A = 0$.
     Let $l_0 = 0$. Noting that l_0 is a lower bound of A, it remains to show that for any $\epsilon > 0$, $l_0 + \epsilon$ is not a lower bound of A (no matter how small ϵ is). It follows by the observation that $a := \frac{\min(\epsilon, 1)}{2}$ satisfies $a \in A$ and $a < l_0 + \epsilon$.
     (Remark: Notice that $\inf A$ exists in $\mathbb{R}$ but does not exist in A.)

   - $\max A = 1$.
     Simply note that $1 \in A$ and $a \leq 1$ for all $a \in A$.

   - $\sup A = 1$.
     It is because for any $B \subseteq \mathbb{R}$, if $\max B$ exists, then $\sup B = \max B$.
     Reason: On the one hand, $\max B$ is an upper bound of B. On the other hand, given any $\epsilon > 0$, $\max B - \epsilon$ fails to be an upper bound of B, because $b_0 := \max B$ satisfies $b_0 \in B$ and $\max B - \epsilon < b_0$.

2. - $\min S$ does not exist.
     Suppose $\min S$ exists. By the definition of "min", $\min S$ is an element in S, so there exists $n_0, m_0 \in \mathbb{N}$ such that $\min S = \frac{1}{n_0} - \frac{1}{m_0}$. But then $\min S > \frac{1}{n_0+1} - \frac{1}{m_0}$. As $\frac{1}{n_0+1} - \frac{1}{m_0} \in S$, this contradicts the minimality of $\min S$. Hence $\min S$ does not exist.

   - $\inf S = -1$.
     Noting that -1 is a lower bound of S, it remains to show that for any $\epsilon > 0$, $-1 + \epsilon$ is not a lower bound of S. Recall that by the Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < \epsilon$.

[1] This solution is adapted from the work by former TAs.

Hence $-1+\frac{1}{n_0} < -1+\epsilon$. Since $-1+\frac{1}{n_0} = \frac{1}{n_0} - \frac{1}{1} \in S$, the result follows.

- $\max S$ does not exists.
  One can use similar reasoning as for min.
  Alternatively, for any $B \subseteq \mathbb{R}$, if we denote $\{x \in \mathbb{R} : -x \in B\}$ by $-B$, then we have: $(-1) \times \max B = \min(-B)$ (either both sides exist or do not exist), because:

$$\begin{cases} b_0 \in B \\ b \leq b_0 \ \forall b \in B \end{cases} \Leftrightarrow \begin{cases} -b_0 \in -B \\ -b_0 \leq b' \ \forall b' \in -B \end{cases}$$

  Observe that $S = -S$.

- $\sup S = 1$.
  One can use similar reasoning as for inf.
  Alternatively, observe that for any $B \subseteq \mathbb{R}$, we have $(-1)\times \sup B = \inf(-B)$ (either both sides exist or do not exist), because:

$$\begin{cases} u_0 \text{ is an upper bound of } B \\ u_0 - \epsilon < b \text{ for some } b \in B \end{cases} \Leftrightarrow \begin{cases} -u_0 \text{ is a lower bound of } -B \\ b' < -u_0 + \epsilon \text{ for some } b' \in -B \end{cases}$$

3. (4 marks)

   For convenience, write $f_1 = f, f_2 = g$, and $f_i(X) = \{f_i(x) : x \in X\}$. Note that for $i = 1, 2$, f_i is bounded above, so the set $f_i(X)$ is bounded above too. By completeness, the supremum for the set $f_i(X)$, denoted by $\sup[f_i(X)]$, exists.

   For each $y \in X$, since $f_i(y) \in f_i(X)$, $f_i(y)$ cannot be greater than the supremum of $f_i(X)$, so $f_i(y) \leq \sup[f_i(X)]$. Adding up, we have $f_1(y) + f_2(y) \leq \sup[f_1(X)] + \sup[f_2(X)]$. This inequality holds for all $y \in X$.

   Therefore, the set $(f_1 + f_2)(X) = \{f_1(x) + f_2(x) : x \in X\}$ is bounded above by the value $\sup[f_1(X)] + \sup[f_2(X)]$, so the supremum of this set cannot be greater than that value. This means

$$\sup[(f_1 + f_2)(X)] \leq \sup[f_1(X)] + \sup[f_2(X)].$$

   Strict inequality can happen. For example, take $X = \{-1, 1\}$, $f_1(x) := x$, $f_2 := -f_1$. Then $f_1(X) = f_2(X) = \{-1, 1\}$, while $(f_1 + f_2)(X) =$

$\{0\}$. So

$$0 = \sup[(f_1 + f_2)(X)] < \sup[f_1(X)] + \sup[f_2(X)] = 1 + 1 = 2.$$

Equality can also happen: Take $X = \{0\}, f_i(x) = x$. Then $f_1(X) = f_2(X) = (f_1 + f_2)(X) = \{0\}$, so

$$0 = \sup[(f_1 + f_2)(X)] = \sup[f_1(X)] + \sup[f_2(X)].$$

We handle inf similarly[2]. Assuming f_i is bounded below function on X so that $\inf[f_i(X)]$ exists, we have, for any $y \in X$,

$$\begin{cases} \inf[f_1(X)] \leq f_1(y) \\ \inf[f_2(X)] \leq f_2(y) \end{cases},$$

so $\inf[f_1(X)] + \inf[f_2(X)]$ is a lower bound of $(f_1 + f_2)(X)$, and consequently

$$\inf[f_1(X)] + \inf[f_2(X)] \leq \inf[(f_1 + f_2)(X)].$$

The first example above gives strict inequality $(-2 < 0)$, while the second example gives equality $(0 = 0)$.

4. (a). Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < \varepsilon$.
   As a corollary of triangle inequality (textbook 2.2.4 Corollary), we have $||x_n| - |x|| \leq |x_n - x|$.
   Hence $||x_n| - |x|| < \varepsilon$ for all $n > N$.
   Therefore, $\lim_{n\to\infty} |x_n| = |x|$.

   (b). Note that $\varepsilon_0 > 0$ since $\alpha < x < \beta$.
   So there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon_0$ for all $n > N$.
   Equivalently, $-\varepsilon_0 < x_n - x < \varepsilon_0$ for all $n > N$.
   Hence for all $n > N$, by the definition of ε_0,

$$\begin{cases} x_n - x < \varepsilon_0 \leq \beta - x \\ -(x - \alpha) \leq -\varepsilon_0 < x_n - x \end{cases}.$$

   This implies $\alpha < x_n < \beta$.

[2] Alternatively, after having the result for sup, one can try the idea in the last part of question 2 to get the result for inf.

5. A is non-empty because $0 \in A$. Also, A is bounded above by $\frac{x-z}{\ell}$. Therefore, $\sup A$ exists. Since $\sup A - 1$ fails to be an upper bound of A, there exists $\overline{n}$ in A such that $\sup A - 1 < \overline{n}$. Since $\overline{n} \in A$, we have $\overline{n} \in \mathbb{N} \cup \{0\}$. Therefore $\overline{n} + 1$ is in $\mathbb{N} \cup \{0\}$ too, and it is greater than the supremum of A, so it cannot be an element in A. As a result,

$$z + \overline{n}\ell \leq x < z + (\overline{n} + 1)\ell,$$

where the first inequality comes from $\overline{n} \in A$ and the second comes from $\overline{n} + 1 \notin A$.

(This also implies $\overline{n} = \max A$.)

Finally, for $\overline{m} := \overline{n} + 1$, the inequalities above give

$$x < z + \overline{m}\ell = z + \overline{n}\ell + \ell < z + \overline{n}\ell + (y - x) \leq x + (y - x) = y.$$

6. (3 marks) If $x \geq 0$, then $-1 \in A$. Else if $x < 0$, then by the Archimedean property, there is an $N \in \mathbb{N}$ such that $N > -xn$, so $-N \in A$. We see that A is non-empty in both cases. Note that A is bounded above by nx. Therefore, by the completeness property of real number, $\sup A$ exists.

By essentially the same argument as in question 5, A has a largest element which we denote by $\overline{\kappa}$. This means

$$\frac{\overline{\kappa}}{n} \leq x < \frac{\overline{\kappa} + 1}{n},$$

which implies

$$x < \frac{\overline{\kappa} + 1}{n} = \frac{\overline{\kappa}}{n} + \frac{1}{n} < \frac{\overline{\kappa}}{n} + (y - x) \leq x + (y - x) = y.$$

Note that $\frac{\overline{\kappa}+1}{n} \in \mathbb{Q}$. Done.