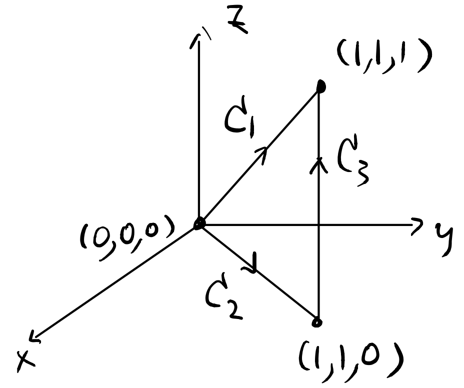
eg34: let $f(x,y,z) = x - 3y^2 + z$ (again)

C_1, C_2, C_3 are line segments as in the figure



We already did $\int_{C_1} f\,ds = 0$ (eg32)

One can similarly calculate

$$\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2} \quad (\text{ex!})$$

(for instance $\int_{C_3} f\,ds = \int_0^1 (1 - 3(1)^2 + t)\,dt$)
(what parametrization?)

The observation is $\int_{C_1} f\,ds = 0 \neq \int_{C_2 \cup C_3} f\,ds$

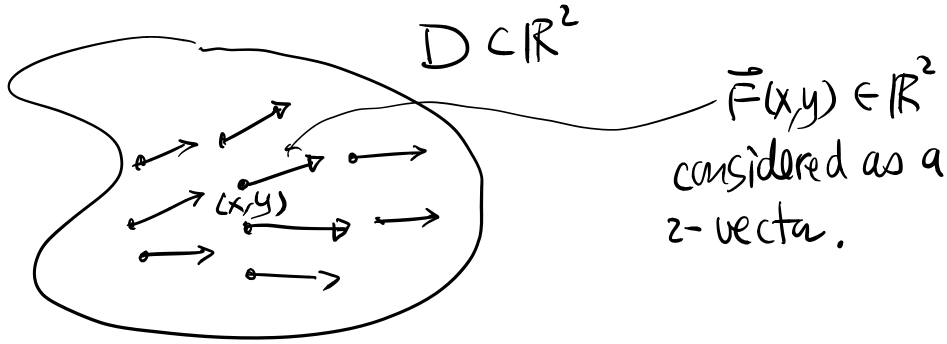
even C_1 & $C_2 \cup C_3$ have the same beginning and end points!

(Remark: different from 1-variable calculus)

Conclusion: Line integral of a function depends, not only on the end points, but also the path.

Vector Fields

Def 10 : Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a vector field on D is a mapping $\vec{F} : D \to \mathbb{R}^2$ or $\mathbb{R}^3$ respectively



In component form :

$\mathbb{R}^2$: $\quad \vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

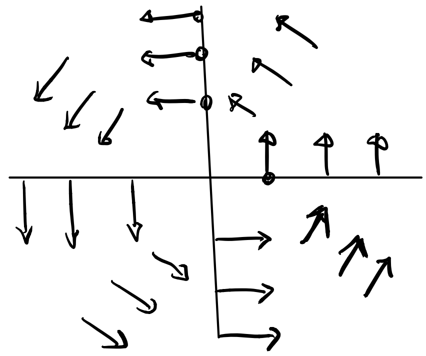
$\mathbb{R}^3$: $\quad \vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + L(x,y,z)\hat{k}$

where M, N, L are functions on D called the components of $\vec{F}$.

eg 35 $\quad \vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ (in polar coordinates)

Properties of $\vec{F}$



(i) $|\vec{F}(x,y)| = 1$

(ii) $\vec{F} \perp \vec{r}(x,y) = x\hat{i} + y\hat{j}$

$= r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

(Ex: Sketch $\vec{F}(x,y) = x\hat{i} + y\hat{j}$)

#

<u>eg36</u> (<u>Gradient vector field of a function</u>)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) \overset{\text{def}}{=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i} + y\hat{j}$$

$$= \vec{r}(x,y) \; (= \vec{r}\,)$$

(ii) $f(x,y,z) = x$

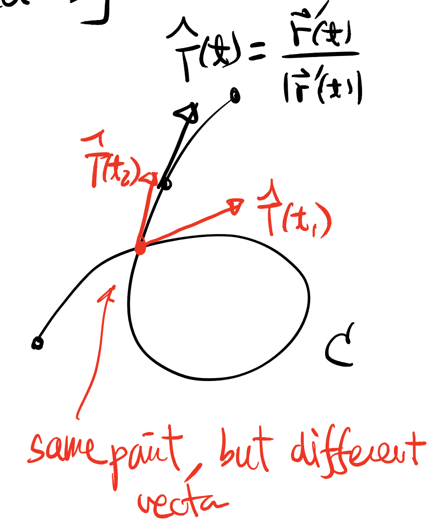
$$\vec{\nabla} f(x,y,z) \overset{\text{def}}{=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (1,0,0) = \hat{i}$$

<u>eg37</u> (Vector field along a curve)

Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\vec{r} : [a,b] \to \mathbb{R}^2$$

$$t \mapsto (x(t), y(t)) = \vec{r}(t)$$



<u>Recall</u>: $\hat{T}$ = unit tangent vector field along C $= \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Note: this $\hat{T}$ defined only on C (for a general curve), but <u>not</u> outside C.

(vector field along a curve may not come from a vector field on a region.)

<u>Remark</u>: for eg37.

If we use $ds = |\vec{r}\,'(t)|\,dt$, then

$$\hat{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds} \quad \text{(by Chain rule)}$$

(if s is a function)

where "arc-length s" is defined (up to an additive constant) by

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(t)|\,dt,$$

A parametrization of a curve C by arc-length s is called <u>arc-length parametrization</u>:

$\vec{r}(s)$ = arc-length parametrization

$$\Rightarrow \quad \left|\frac{d\vec{r}}{ds}(s)\right| = 1$$

Def 11 A vector field is defined to be continuous / differentiable / C^k if the component functions are.

eg38: $\vec{F}(x,y) = \vec{r}(x,y) = x\hat{i} + y\hat{j}$ is C^∞

$\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ is not continuous in $\mathbb{R}^2$ (but continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$)

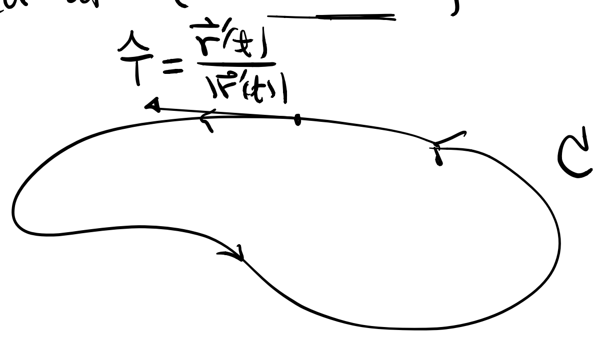
Line integral of vector field

Def 12: Let C be a curve with "orientation" given by a parametrization $\vec{r}(t)$ with $\vec{r}'(t) \neq 0$, $\forall t$. Define the line integral of a vector field $\vec{F}$ along C to be

$$\int_C \vec{F} \cdot \hat{T}\, ds$$

where $\hat{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is the unit tangent vector field along C.

(i.e. C is oriented in the direction of $\vec{r}'(t)$ or $\hat{T}$ at every point)



Note: If $\vec{r}:[a,b] \to \mathbb{R}^n$ ($n=2$ or 3) then

$$\int_C \vec{F}\cdot\hat{T}\,\underbrace{ds} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\cdot|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot \underbrace{\vec{r}\,'(t)\,dt} \longrightarrow d\vec{r}$$

$\therefore$ naturally, we denote $\boxed{d\vec{r} = \hat{T}\,ds}$

and $\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}}$

eg 38: $\vec{F}(x,y,z) = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$C: \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}, \quad 0\le t\le 1$

Then $d\vec{r} = (2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k})\,dt$

and $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_C (z\hat{i} + xy\hat{j} - y^2\hat{k})\cdot(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k})\,dt$$

$$= \int_0^1 (\sqrt{t}\,\hat{i} + (t^2)(t)\hat{j} - (t)^2\hat{k})\cdot(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k})\,dt$$

$$= \int_0^1 (2t\sqrt{t} + t^3 - \frac{t^{3/2}}{2})\,dt = \frac{17}{20} \text{ (check!)}$$

#

In components form:

Line integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$$C: \vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$$

can be expressed as

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right) dt$$

$$= \int_a^b (Mg' + Nh')\,dt$$

$$\left(\text{more explicitly: } \int_a^b [M(g(t), h(t))\,g'(t) + N(g(t), h(t))\,h'(t)]\,dt\right)$$

Note that, $\begin{cases} x = g(t) \\ y = h(t) \end{cases}$

$$\Rightarrow \begin{cases} dx = g'(t)\,dt \\ dy = h'(t)\,dt \end{cases}$$

$$\therefore \boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b M\,dx + N\,dy}$$

Similarly, for 3-dim.

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b M\,dx + N\,dy + L\,dz}$$

$$(\text{for } \vec{F} = M\hat{i} + N\hat{j} + L\hat{k})$$

Another way to justify the notation:

$\vec{r} = (x, y, z)$ the position vector

$\Rightarrow \boxed{d\vec{r} = (dx, dy, dz)}$ (naturally notation)

Then $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (M, N, L)\cdot(dx, dy, dz)$

$= \int_C M\,dx + N\,dy + L\,dz$.

eg 39: Evaluate $I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le 2\pi)$

$= (\cos t, \sin t, t)$

Soln: $d\vec{r} = (-\sin t, \cos t, 1)\,dt$

$\Rightarrow I = \int_0^{2\pi} [(-\sin t)(-\sin t) + t\cos t + 2\cos t\,(1)]\,dt$

$= \cdots = \pi$ (check!)

#

Physics

(1) $\vec{F}$ = Force field

C = oriented curve

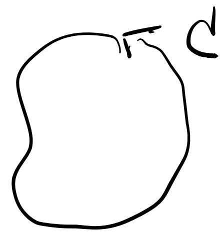
then $$\boxed{W = \int_C \vec{F} \cdot \hat{T}\, ds}$$

is the workdone in moving an object along C.

(2) $\vec{F}$ = velocity vector field of fluid

C = oriented curve

Then $$\boxed{\text{Flow} = \int_C \vec{F} \cdot \hat{T}\, ds}$$



Flow along the curve C.

If C is "closed", the flow is also called a circulation.

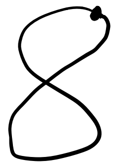
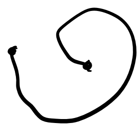
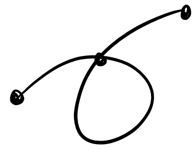
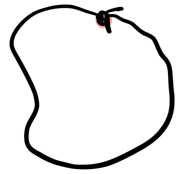
Def 13: A curve is said to be

(i) simple if it does not intersect with itself except possibly at end points.

(ii) closed if starting point = end point.

(iii) simple closed curve if it is both simple and closed.

Note:

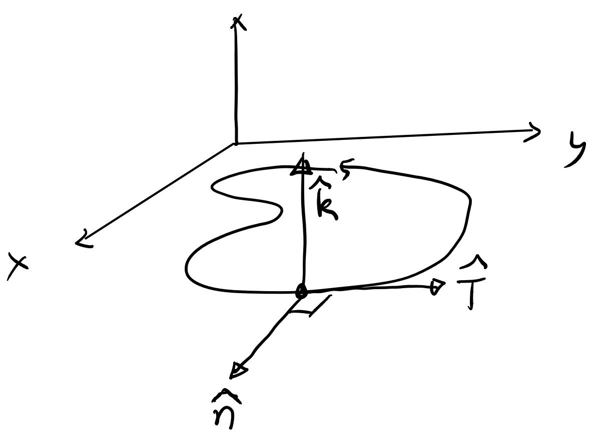
				
simple	NO	Yes	NO	Yes
closed	Yes	NO	NO	Yes

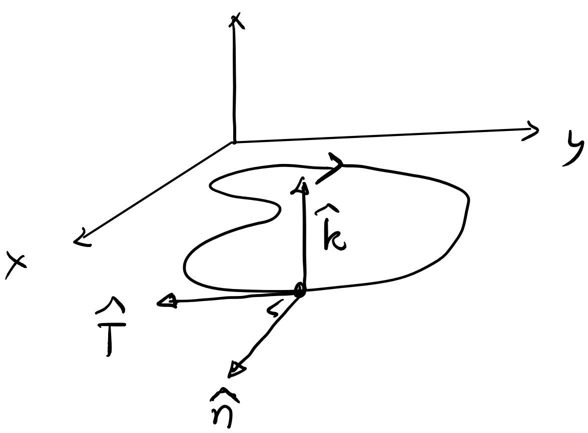
(3) $\vec{F}$ = velocity of fluid

C = oriented <u>plane</u> curve ($C \subset \mathbb{R}^2$)

with parametrization $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$

$\hat{n}$ = <u>outward-pointing unit normal (vector)</u> to the curve C





$$\boxed{\hat{n} = \hat{T} \times \hat{k}}$$

if C is of anti-clockwise orientation

$$\boxed{\hat{n} = -\hat{T} \times \hat{k}}$$

if C is of clockwise orientation.

Formula for $\hat{n}$ (wrt the parametrization $\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}$)

Recall $\hat{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{x'(t)\hat{i}+y'(t)\hat{j}}{|\vec{r}\,'(t)|}$

(in arc-length parametrization : $\hat{T} = \dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j}$)

<u>Anti-clockwise</u> :

$$\hat{n} = \hat{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{x'}{|\vec{r}\,'|} & \dfrac{y'}{|\vec{r}\,'|} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \hat{n} = \frac{y'(t)\hat{i} - x'(t)\hat{j}}{|\vec{r}\,'(t)|} \quad \left(\text{or } \hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \right)$$

<u>Clockwise</u> :

$$\hat{n} = \frac{-y'(t)\hat{i} + x'(t)\hat{j}}{|\vec{r}\,'(t)|} \quad \left(\text{or } \hat{n} = -\frac{dy}{ds}\hat{i} + \frac{dx}{ds}\hat{j} \right)$$

$$\boxed{\text{Flux of } \vec{F} \text{ across } C \overset{\text{def}}{=} \int_C \vec{F}\cdot\hat{n}\, ds}$$

(amount of fluid getting out of the closed curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is <u>anti-clockwise</u> parametrization of C (<u>closed</u> curve)

Then

Flux of $\vec{F}$ across C

$$= \oint_C (M\hat{i} + N\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$$

$$= \oint_C M\,dy - N\,dx$$

<u>Remark</u>:

- $\ointctrclockwise$: curve is <u>closed</u> & in <u>anti-clockwise</u> direction
- $\ointclockwise$ = curve is <u>closed</u> & in <u>clockwise</u> direction (not a common notation)
- But in some books, only "$\oint$" is used, <u>NO</u> arrow, Then one needs to determine the orientation from the context.
- <u>Convention</u> : If no orientation is mentioned, "$\oint$" without arrow means <u>anti-clockwise orientation</u> (<u>positive orientation</u>)