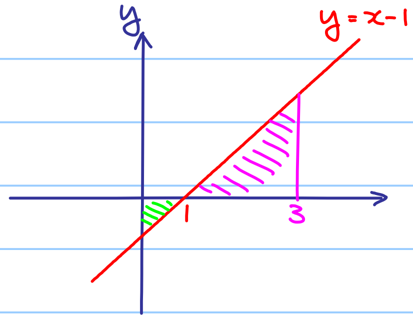
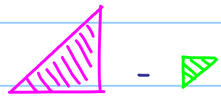
e.g. (NOT area, but signed area)



$$\int_0^1 x-1\,dx = \left[\frac{x^2}{2} - x\right]_0^1 = -\frac{1}{2}$$

$$\int_1^3 x-1\,dx = \left[\frac{x^2}{2} - x\right]_1^3 = 2$$

$$\int_0^3 x-1\,dx = \left[\frac{x^2}{2} - x\right]_0^3 = \frac{3}{2}$$



(Cancellation)

e.g. $\displaystyle\int_{-2}^{3} |x-1|\,dx$

Recall : We can rewrite $\quad |x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -(x-1) & \text{if } x < -1 \end{cases}$

$$\int_{-2}^{3} |x-1|\,dx = \int_{-2}^{1} |x-1|\,dx + \int_{1}^{3} |x-1|\,dx$$

$$= \int_{-2}^{1} -(x-1)\,dx + \int_{1}^{3} x-1\,dx$$

$\vdots$

Ex : $= \frac{9}{2} + 2 = \frac{13}{2}$

e.g. (Fundamental Theorem of Calculus)

Find $\frac{dF}{dx}$ if

a) $F(x) = \int_0^x e^{\cos t}\, dt$, b) $F(x) = \int_0^{x^2} e^{\cos t}\, dt$, c) $F(x) = \int_x^{x^2} e^{\cos t}\, dt$

a) $\frac{dF}{dx} = e^{\cos x}$ (Directly from Fundamental Theorem of Calculus, $f(x) = e^{\cos x}$)

b) $\frac{dF}{dx} = \frac{d}{dx^2} \int_0^{x^2} e^{\cos t}\, dt \cdot \frac{dx^2}{dx}$ (Chain rule)

$= e^{\cos x^2} \cdot 2x$

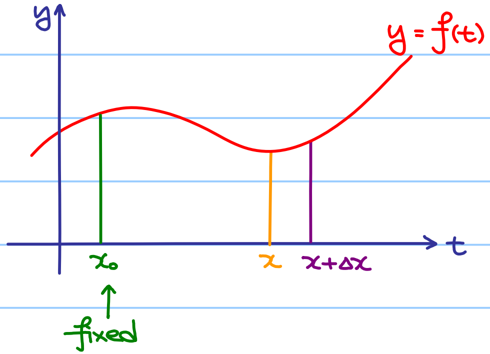
$= 2x e^{\cos x^2}$

c) $\frac{dF}{dx} = \frac{d}{dx} \int_0^{x^2} e^{\cos t}\, dt - \frac{d}{dx} \int_0^{x} e^{\cos t}\, dt$

$= 2x e^{\cos x^2} - \cos x$

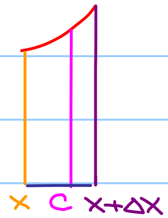
Sketch of the proof Fundamental Theorem of Calculus

Claim: If $F(x) = \int_{x_0}^{x} f(t)\,dt$, $\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x)$, i.e. $F'(x) = f(x)$



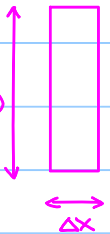
$F(x+\Delta x) - F(x)$

$=$ Area of



By continuity of f, there exists $c \in (x, x+\Delta x)$ such that

$= f(c)$



$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(c)\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(c)$$

$$= \lim_{c \to x} f(c)$$ (As Δx tends to 0 , c tends to x)

$$= f(x)$$ (By continunity of f)

e.g. Find $\displaystyle \lim_{n\to\infty} \underbrace{\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3}}_{n \text{ terms}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$

Note: As $n \to \infty$, it is an infinite sum, i.e. summing infinitely many terms.

Algebraic rule does NOT work !!

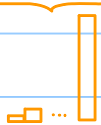
We cannot say: $\displaystyle \lim_{n\to\infty} \frac{1^2}{n^3} = \lim_{n\to\infty} \frac{2^2}{n^3} = \cdots = \lim_{n\to\infty} \frac{n^2}{n^3} = 0$

$\displaystyle \therefore \lim_{n\to\infty} \frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3} = 0$

Recall : $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f\left(a+(b-a)\frac{i}{n}\right)\cdot\frac{b-a}{n} = \int_a^b f(x)\,dx$

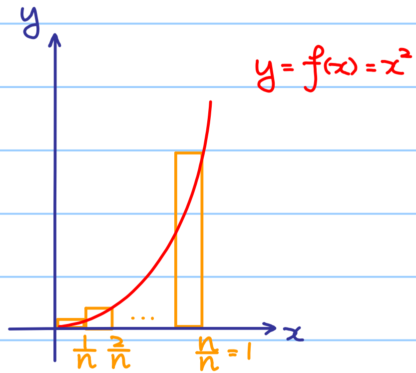
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{n^3} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{n^2}\,\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$



In this case,

$a = 0$, $b = 1$,



$$= \int_0^1 f(x)\,dx$$

$$= \frac{1}{3}$$

Roughly, $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$

$$\int_a^b f(x)\,dx$$

e.g. Find $\lim_{n\to\infty} \frac{1}{n}\left(e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \cdot \frac{1}{n}$$

$$= \int_0^1 e^x \, dx$$

$$= \left[e^x\right]_0^1$$

$$= e^1 - e^0$$

$$= e - 1$$

Definite Integral Using Substitution

$$\int_a^b f(u(x)) \cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

e.g. $\int_0^1 8x(x^2+1)\,dx$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

Similar to indefinite integration

$= \int_0^1 8(x^2+1)\; x\,dx$

when $x=0$, $u=1$

$x=1$, $u=2$

New! Don't forget!

$= \int_1^2 8u\; \frac{1}{2}du$

Caution!

$= \int_1^2 4u\,du$

$= \left[2u^2\right]_1^2$

$= 6$

Remark:

Some may write :

Still 0 and 1

$$\int_0^1 8x(x^2+1)\,dx = \int_0^1 4(x^2+1)\,d(x^2+1)$$ (as $d(x^2+1) = 2x\,dx$)

$$= \left[2(x^2+1)\right]_0^1$$

$$= 6$$

(Just the same result !)

e.g. $\displaystyle\int_{e}^{e^2} \frac{1}{x\ln x}\,dx$

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

when $x = e$, $u = 1$

$x = e^2$, $u = 2$

$= \displaystyle\int_{1}^{2} \frac{1}{u}\,du$

$= [\ln u]_1^2$

$= \ln 2 - \cancelto{0}{\ln 1}$

$= \ln 2$

More on Substitution

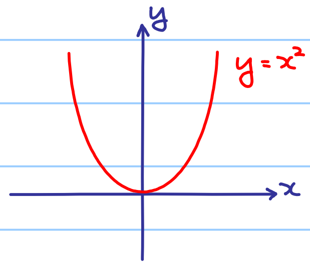
Recall :

A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is said to be

- even if $f(-x)=f(x)$ for all $x\in\mathbb{R}$

  e.g. x^2, $\cos x$, $|x|$

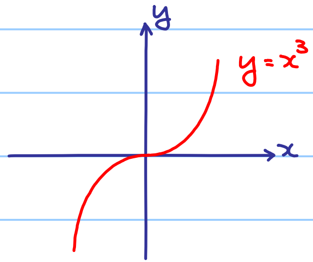
  property : the graph is symmetric along y-axis.



- odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$

  e.g. x^3 , $\sin x$

  property : the graph is symmetric about the origin

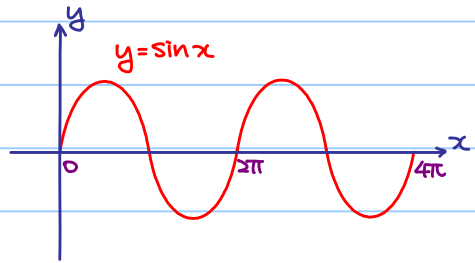


- periodic if there exists $T>0$ such that $f(x)=f(x+T)$ for all $x \in \mathbb{R}$

If $T>0$ is the least positive real number with the above property, T is called the period.

e.g. $\sin x$, $\cos x$, $\tan x$

property : the graph is repeating again and again



period of $\sin x$, $\cos x$ $= 2\pi$

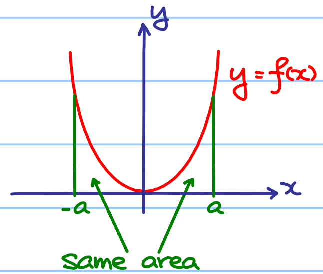
period of $\tan x = \pi$

Suppose f is an even function and $a>0$, prove that $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$\parallel ?$

$$\int_{0}^{a} f(x)\,dx$$



$$\int_{-a}^{0} f(x)\,dx$$ Let $y=-x$

$dy=-dx$

$$= \int_{a}^{0} -f(y)\,dy$$ When $x=0,\ y=0$

$f(-y)=f(y)$

$$= \int_{0}^{a} f(y)\,dy$$ $x=-a,\ y=a$

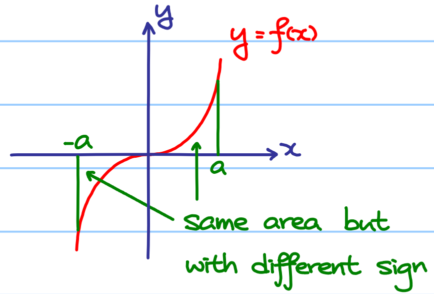
$$= \int_{0}^{a} f(x)\,dx$$ (dummy variable)

e.g. $\int_{-4}^{4} |x|\,dx = 2\int_{0}^{4} |x|\,dx = 2\int_{0}^{4} x\,dx = 2\left[\frac{x^2}{2}\right]_0^4 = 16$

Suppose f is an odd function and $a>0$, prove that $\int_{-a}^{a} f(x)\,dx = 0$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\overset{?}{=} -\int_{0}^{a} f(x)\,dx$$



$$\int_{-a}^{0} f(x)\,dx$$

Let $y=-x$

$dy=-dx$

When $x=0$, $y=0$

$x=-a$, $y=a$

$$= \int_{a}^{0} -f(-y)\,dy$$

$f(-y)=-f(y)$

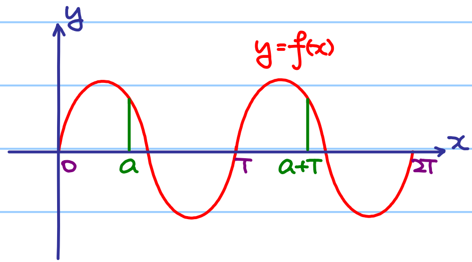
$$= \int_{0}^{a} -f(y)\,dy$$

$$= -\int_{0}^{a} f(x)\,dx \quad \text{(dummy variable)}$$

e.g. $\int_{-\pi}^{\pi} e^{\sin x}\,dx = 0$ $\because$ $e^{\sin x}$ is an odd function.

Suppose f is a periodic function with period $T>0$ and $a \in \mathbb{R}$,

prove that $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$



$$\int_a^{a+T} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

$$\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx \qquad \int_T^{a+T} f(x)\,dx \overset{?}{=} \int_0^a f(x)\,dx$$

$$\int_T^{a+T} f(x)\,dx$$

Let $y = x - T$

$dy = dx$

When $x = T$, $y = 0$

$x = a+T$, $y = a$

$$= \int_0^a f(y+T)\,dy$$

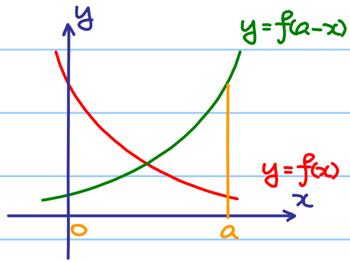
$f(y+T) = f(y)$

$$= \int_0^a f(y)\,dy$$

$$= \int_0^a f(x)\,dx \quad \text{(dummy variable)}$$

e.g. (Similar example)

Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$



$\int_0^a f(a-x)\,dx$

$= \int_a^0 -f(y)\,dy$

$= \int_0^a f(y)\,dy$

$= \int_0^a f(x)\,dx$ (dummy variable)

Let $y = a - x$

$dy = -dx$

When $x = 0$, $y = a$

$x = a$, $y = 0$

Definite Integration Using Integration by Parts

$$\int_a^b u \frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v \frac{du}{dx}\,dx$$

e.g. $$\int_1^e x \ln x\, dx = \int_1^e \ln x \, d\left(\frac{x^2}{2}\right)$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\, d\ln x$$

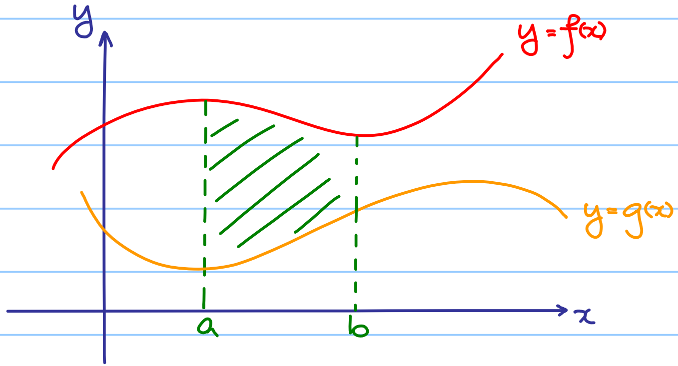
$$= \left(\frac{e^2}{2}\ln e - \cancelto{0}{\frac{1}{2}\ln 1}\right) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

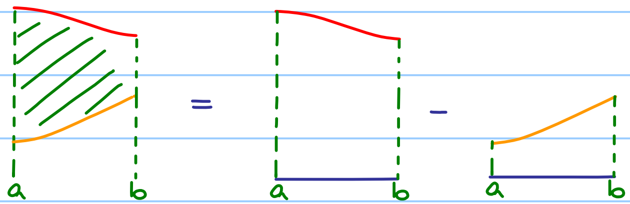
$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Area Between Curves :



Area of shaded region = $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



e.g. Find the area bounded by $y=x^2$ and $y=x^3$.

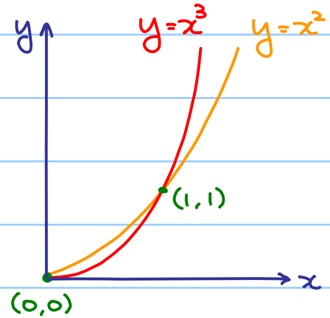
Step 1: Solve $\begin{cases} y=x^2 \\ y=x^3 \end{cases}$

$$x^3=x^2$$

$$x^2(x-1)=0$$

$$x=0 \text{ or } 1$$

(Remark: No need to solve y)



Step 2: Note when $0 \le x \le 1$, $x^3 \le x^2$

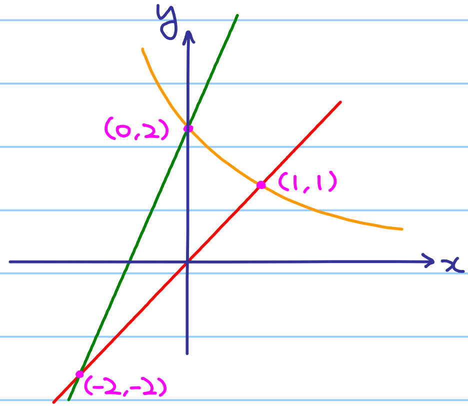
$$\text{Area} = \int_0^1 x^2 - x^3 \, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{12}$$

e.g. Find the area bounded by

$y = f(x) = x$, $y = g(x) = \frac{2}{x+1}$ and $y = h(x) = 2x+2$



Area $= \int_{-2}^{0} h(x) - f(x)\, dx + \int_{0}^{1} g(x) - f(x)\, dx$

Ex : ⋮

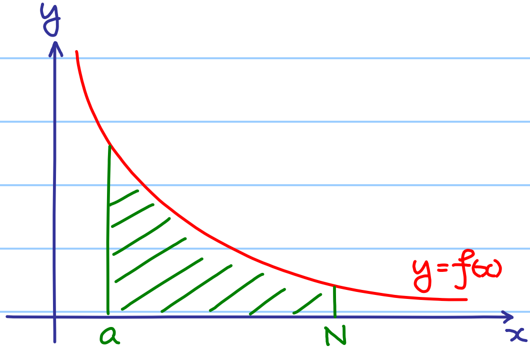
Ans : $= 2 + (-\frac{1}{2} + \ln 4)$

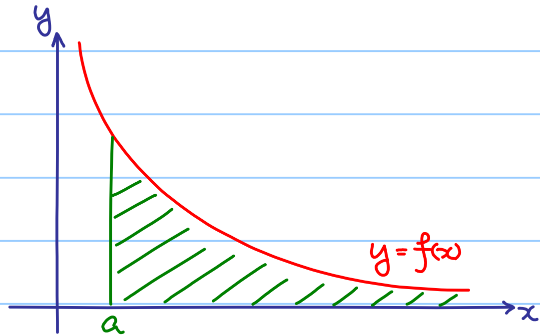
$= \frac{3}{2} + \ln 4$

Improper Integrals :

Question : Find the area of the unbounded region ?

Idea :



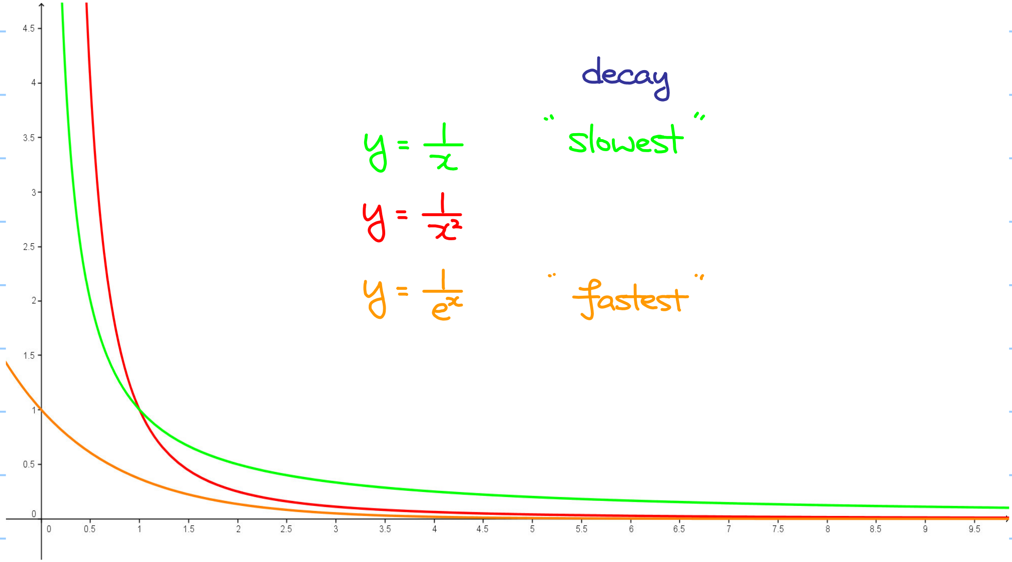


$\int_a^N f(x)dx$ $\rightsquigarrow$ Area of the unbounded region

$= \lim_{N \to +\infty} \int_a^N f(x)dx$ (if it exists)

We denote it by $\int_a^{+\infty} f(x)\,dx$

e.g.



decay

$y = \frac{1}{x}$ "slowest"

$y = \frac{1}{x^2}$

$y = \frac{1}{e^x}$ "fastest"

① $\lim_{N\to+\infty} \int_1^N \frac{1}{x}\,dx = \lim_{N\to+\infty} [\ln x]_1^N = \lim_{N\to+\infty} \ln N = +\infty$ (i.e. limit does NOT exist)

② $\lim_{N\to+\infty} \int_1^N \frac{1}{x^2}\,dx = \lim_{N\to+\infty} [-\frac{1}{x}]_1^N = \lim_{N\to+\infty} 1 - \frac{1}{N} = 1$

③ $\lim_{N\to+\infty} \int_1^N \frac{1}{e^x}\,dx = \lim_{N\to+\infty} [-\frac{1}{e^x}]_1^N = \lim_{N\to+\infty} -\frac{1}{e^N} + \frac{1}{e}$

Observation : $\lim_{x\to+\infty} f(x) = 0$ does NOT guarantee $\lim_{N\to+\infty} \int_a^N f(x)\,dx$ exists.

e.g. Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)}\,dx$

Note : $(x+1)(3x+2)$ is a polynomial of degree 2.

$\frac{1}{(x+1)(3x+2)}$ decays as "fast" as $\frac{1}{x^2}$.

$$\lim_{N\to+\infty} \int_0^N \frac{1}{(x+1)(3x+2)}\,dx = \lim_{N\to+\infty} \int_0^N \frac{-1}{x+1} + \frac{3}{3x+2}\,dx$$

$$= \lim_{N\to+\infty} \left[-\ln|x+1| + \ln|3x+2|\right]_0^N$$

$$= \lim_{N\to+\infty} \ln\left|\frac{3N+2}{N+1}\right| - \ln 2$$

$$= \ln 3 - \ln 2$$

e.g. Find $\int_0^{+\infty} xe^{-2x}\,dx$

$$\lim_{N\to+\infty}\int_0^N xe^{-2x}\,dx$$

$$=\lim_{N\to+\infty}\int_0^N x\,d\left(-\frac{1}{2}e^{-2x}\right)$$

$$=\lim_{N\to+\infty}\left[-\frac{1}{2}xe^{-2x}\right]_0^N-\int_0^N -\frac{1}{2}e^{-2x}\,dx$$

$$=\lim_{N\to+\infty}\left[-\frac{1}{2}xe^{-2x}\right]_0^N+\left[-\frac{1}{4}e^{-2x}\right]_0^N$$

go to 0 when $N\to+\infty$

$$=\lim_{N\to+\infty} -\frac{1}{2}Ne^{-2N}-\frac{1}{4}e^{-2N}+\frac{1}{4}$$

$$=\frac{1}{4}$$