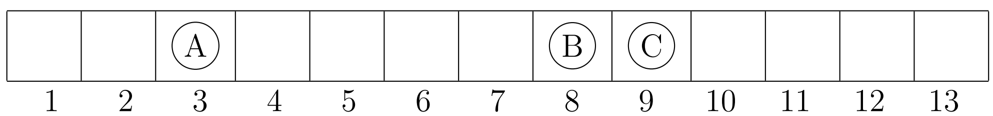
MATH4250 Game Theory

Problem. A game is played on a game board consisting of a line of squares labeled $1, 2, 3, \ldots$ from left to right. Three coins A, B, C are placed on the squares and at any time each square can be occupied by at most one coin. A move consists of taking one of the coins and moving it to a square with a small number so that coin A occupies a square with a number smaller than coin B and coin B occupies a square with a number smaller than coin C. The game ends when there is no possible move, that is coins A, B, C occupy at square number $1, 2, 3$ respectively, and the player who makes the last move wins. Let (x, y, z), where $1 \leq x < y < z$, be the position of the game that coins A, B, C are at squares labeled x, y, z respectively. The position $(3, 8, 9)$ is shown below.



Examples of legal moves from position $(3, 8, 9)$ are $(1, 8, 9)$, $(3, 4, 9)$ and $(3, 5, 9)$. One cannot move coin C from position $(3, 8, 9)$. Define

$$g(x, y, z) = (x-1) \oplus (z-y-1), \text{ for } 1 \leq x < y < z$$

where $\oplus$ denotes nim-sum.

(a) Prove that $g(x, y, z)$ is the Sprague-Grundy function of the game. (All properties of $\oplus$ can be used without proof.)

(b) Find all winning moves from the positions $(6, 13, 25)$ and $(23, 56, 63)$.

Solution.

1. We will use the following property of nim-sum. If $m < a \oplus b$, then there exists $k < a$ such that $k \oplus b = m$, or there exists $k < b$ such that $a \oplus k = m$.
   Now we prove that $g(x, y, z) = (x-1) \oplus (z-y-1)$ is the Sprague-Grundy function.
   For any $m < g(x, y, z) = (x-1) \oplus (z-y-1)$, by the above property of nim-sum, there exists $x' < x$ such that $(x', y, z) \in F(x, y, z)$ and $g(x', y, z) = (x'-1) \oplus (z-y-1) = m$, or there exists $z' < z$ such that $(x, y, z') \in F(x, y, z)$ and $g(x, y, z') = (x-1) \oplus (z'-y-1) = m$.
   For any $(x', y', z') \in F(x, y, z)$, either $x'-1 < x-1$ and $z'-y'-1 = z-y-1$ or $x'-1 = x-1$ and $z'-y'-1 \neq z-y-1$. Thus we must have $(x'-1) \oplus (z'-y'-1) \neq (x-1) \oplus (z-y-1)$.
   Therefore $g(x, y, z)$ is the Sprague-Grundy function of the game.

2. The set of P-position is $P = \{(x, y, z) : g(x, y, z) = 0\} = \{(x, y, z) : x = z - y\}$. The winning moves are
   $(6, 13, 25)$: $(6, 13, 19)$
   $(23, 56, 63)$: $(7, 56, 63)$, $(23, 40, 63)$.