Suggested Solution to Mid-term exam

1. (a)(3') We have $12 = 1100_2$, $23 = 10111_2$, $29 = 11101_2$.

Now

$$\begin{array}{rr} & 1100_2 \\ & 10111_2 \\ \oplus & 11101_2 \\ \hline & 110_2 \end{array}$$

Thus $12 \oplus 23 \oplus 29 = 110_2 = 6$

(b)(3') From (a) we can move $1100_2 \to 1010_2 = 10$, or $10111_2 \to 10001_2 = 17$, or $11101_2 \to 11011_2 = 27$.

Hence there are three winning moves: $(10, 23, 29)$, $(12, 17, 29)$, $(12, 23, 27)$.

2. (a)(3') $g(3,1) = 3$, $g(2,2) = 0$, $g(3,2) = 6$.

(b)(3') The set of P-positions: $P = \{(x,y) : x = y,\ x, y \in \mathbb{N}\}$.

(c)(4') Pf. (i) The only terminal position is $(0,0)$ and $(0,0) \in P$.

(ii) Take any position $p \in P$, i.e $p = (x, x)$, $x > 0$. And it can be moved to a position $q = (x', y')$. There are three cases: $x' < x$, $y' = x$; $x' = x$, $y' < x$; $x' < x$, $y' > x$. In a word, $x' \neq y'$. Thus any move from any position $p \in P$ reaches a position $q \notin P$.

(iii) For any position $q \notin P$, i.e $q = (x, y)$, $x \neq y$. If $x > y$, then the next player may remove $(x-y)$ chips from the first pile to reach $p = (y, y) \in P$. If $x < y$, then the next player may remove $(y-x)$ chips from the second pile to reach $p = (x, x) \in P$. Thus, for any $q \notin P$, there exists a move from q reaching a position $p \in P$.

Therefore, $P = \{(x,y) : x = y,\ x, y \in \mathbb{N}\}$ is the set of P-positions.

3. (a)(3') According to the lecture notes, $g_1(x)=x$, $g_2(x)\equiv x \pmod 7$, $g_3(x)=\min\{k\in\mathbb{N}: 2^k>x\}$.

Hence $g_1(6)=6$, $g_2(12)\equiv 12 \pmod 7 = 5$, $g_3(13)=\min\{k\in\mathbb{N}: 2^k>13\}=4$.

(b)(2') $g(6,12,13)=g_1(6)\oplus g_2(12)\oplus g_3(13)=6\oplus 5\oplus 4 = 110_2\oplus 101_2 \oplus 100_2 = 111_2 = 7$.

(c)(5') Let the winning move of G from the position $(6,12,13)$ be (x,y,z).

We can make a move in exactly one of the Game 1, 2, 3 such that

Game 1: $110_2 \to 001_2$ i.e $g_1(x)=001_2=1$

Game 2: $101_2 \to 10_2$ i.e $g_2(y)=10_2=2$

Game 3: $100_2 \to 11_2$ i.e $g_3(z)=11_2=3$

For $g_1(x)=1$, we may take $x=1$; For $g_2(y)=2$, we may take $y=9$;

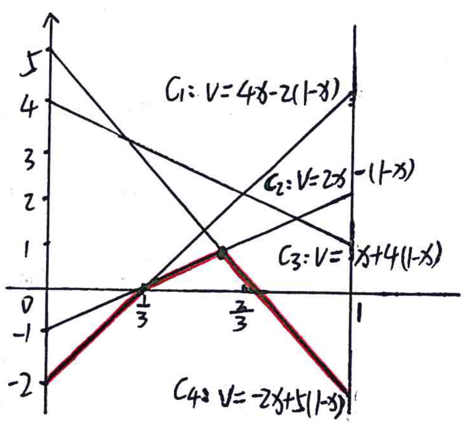
For $g_3(z)=3$, we may take $z=6,5,4$.

Hence all winning moves are: $(1,12,13)$, $(6,9,13)$, $(6,12,6)$, $(6,12,5)$, $(6,12,4)$.

4. (a)(4') $\begin{pmatrix} 4 & 2 & 1 & 3 & -2 \\ 2 & 1 & -1 & 4 & -3 \\ -2 & -1 & 4 & 0 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 2 & 1 & -2 \\ 2 & 1 & -1 & -3 \\ -2 & -1 & 4 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 2 & 1 & -2 \\ -2 & -1 & 4 & 5 \end{pmatrix}$

$A'=\begin{pmatrix} 4 & 2 & 1 & -2 \\ -2 & -1 & 4 & 5 \end{pmatrix}$

(b)(4') For A', by drawing the lower envelope, the maximum point of the lower envelope is the intersection point of C_2 and C_4. By solving



$$\begin{cases} C_2: v=2x-(1-x) \\ C_4: v=-2x+5(1-x) \end{cases}$$

$x=\frac{3}{5}$, $v=\frac{4}{5}$.

For the minimax strategy: $\begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} y_2 \\ y_4 \end{pmatrix}=\begin{pmatrix} \frac{4}{5} \\ \frac{4}{5} \end{pmatrix} \Rightarrow y_2=\frac{7}{10},\ y_4=\frac{3}{10}$.

Hence the value of the game is $\frac{4}{5}$, a maximin strategy for the row player is $(\frac{3}{5},0,\frac{2}{5})$, a minimax strategy for the column player is $(0,\frac{7}{10},0,0,\frac{3}{10})$.

5.(8) Add $k=3$ to every entry to get $\begin{pmatrix} 5 & 2 & 4 \\ 4 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

Applying simplex algorithm, we have

$$
\begin{array}{c|ccc|c}
 & y_1 & y_2 & y_3 & -1 \\
\hline
x_1 & 5^* & 2 & 4 & 1 \\
x_2 & 4 & 0 & 3 & 1 \\
x_3 & 1 & 3 & 2 & 1 \\
\hline
-1 & 1 & 1 & 1 & 0
\end{array}
\longrightarrow
\begin{array}{c|ccc|c}
 & x_1 & y_2 & y_3 & -1 \\
\hline
y_1 & \frac{1}{5} & \frac{2}{5} & \frac{4}{5} & \frac{1}{5} \\
x_2 & -\frac{4}{5} & -\frac{8}{5} & -\frac{1}{5} & \frac{1}{5} \\
x_3 & -\frac{1}{5} & \frac{13}{5}^* & \frac{6}{5} & \frac{4}{5} \\
\hline
-1 & -\frac{1}{5} & \frac{3}{5} & \frac{1}{5} & -\frac{1}{5}
\end{array}
\longrightarrow
\begin{array}{c|ccc|c}
 & x_1 & x_3 & y_3 & -1 \\
\hline
y_1 & \frac{3}{13} & -\frac{2}{13} & \frac{8}{13} & \frac{1}{13} \\
x_2 & -\frac{12}{13} & \frac{8}{13} & \frac{7}{13} & \frac{9}{13} \\
y_2 & -\frac{1}{13} & \frac{5}{13} & \frac{6}{13} & \frac{4}{13} \\
\hline
-1 & -\frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} & -\frac{5}{13}
\end{array}
$$

The independent variables are $x_2, x_4, x_5, y_3, y_4, y_6$ and the basic variables are $x_1, x_3, x_6, y_1, y_2, y_5$. The basic solution is

$$x_2 = x_4 = x_5 = 0,\ x_1 = \frac{2}{13},\ x_3 = \frac{3}{13},\ x_6 = \frac{1}{13}.$$

$$y_3 = y_4 = y_6 = 0,\ y_1 = \frac{1}{13},\ y_2 = \frac{4}{13},\ y_5 = \frac{9}{13}.$$

The optimal value is $d = \frac{5}{13}$. Therefore a maximin strategy for the row player is

$$\vec{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{13}{5}\left(\frac{2}{13}, 0, \frac{3}{13}\right) = \left(\frac{2}{5}, 0, \frac{3}{5}\right).$$

A minimax strategy for the column player is

$$\vec{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{13}{5}\left(\frac{1}{13}, \frac{4}{13}, 0\right) = \left(\frac{1}{5}, \frac{4}{5}, 0\right).$$

The value of the game is

$$v = \frac{1}{d} - k = \frac{13}{5} - 3 = -\frac{2}{5}.$$

6. (a)(4) The game matrix is

$$
\begin{array}{c|ccccc}
R \backslash C & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 2 & 1 & 1 & 1 & 1 \\
2 & 0 & 2 & 1 & 1 & 1 \\
3 & 0 & 0 & 2 & 1 & 1 \\
4 & 0 & 0 & 0 & 2 & 1 \\
5 & 0 & 0 & 0 & 0 & 2
\end{array}
\quad \text{i.e. } A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.
$$

Assume that Cathy has minimax strategy with positive weight in each entry.

By the principle of indifference, we have that the maximin strategy $\vec{p} = (p_1, \cdots, p_5)$ for Ronald satisfies $\sum_{i=1}^{5} p_i a_{ij} = v,\ j = 1, \cdots, 5$, where v is the value of the game. i.e.

$$\begin{cases} 2p_1 = v & ① \\ p_1 + 2p_2 = v & ② \\ p_1 + p_2 + 2p_3 = v & ③ \\ p_1 + p_2 + p_3 + 2p_4 = v & ④ \\ p_1 + p_2 + p_3 + p_4 + 2p_5 = v & ⑤ \end{cases}$$

By ① and ②, we have $p_1 = 2p_2$; by ② and ③, we have $p_2 = 2p_3$; by ③ and ④, we have $p_3 = 2p_4$; by ④ and ⑤, we have $p_4 = 2p_5$.

Hence $p_1 = 2p_2 = 2^2 p_3 = 2^3 p_4 = 2^4 p_5$.

Since $\sum_{i=1}^{5} p_i = 1$, i.e $p_5 \cdot (2^4 + 2^3 + 2^2 + 2 + 1) = 1 \Rightarrow p_5 = \frac{1}{2^5 - 1} = \frac{1}{31}$.

So $\vec{p} = (\frac{16}{31}, \frac{8}{31}, \frac{4}{31}, \frac{2}{31}, \frac{1}{31})$, $v = \frac{32}{31}$.

Similarly, by principle of indifference, we can get the minimax strategy is $\vec{q} = (\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31})$

(b)(4) Assume that the column player has minimax strategy with positive weight in each entry.

By the principle of indifference, the maximin strategy $\vec{p} = (p_1, p_2, \cdots p_n)$ satisfies

$\sum_{i=1}^{n} p_i a_{ij} = v$, $j = 1, 2, \cdots, n$, where v is the value of the game. i.e.

$$\begin{cases} (a+1)p_1 + p_2 + p_3 + \cdots + p_n = v & ① \\ ap_1 + (a+1)p_2 + p_3 + \cdots + p_n = v & ② \\ ap_1 + ap_2 + (a+1)p_3 + \cdots + p_n = v & ③ \\ \cdots & \vdots \\ ap_1 + ap_2 + ap_3 + \cdots + (a+1)p_{n-1} + p_n = v & (n-1) \\ ap_1 + ap_2 + ap_3 + \cdots ap_{n-1} + (a+1)p_n = v & (n) \end{cases}$$

Comparing ① and ②, we have $p_1 = ap_2$; Comparing ② and ③, we have $p_2 = ap_3$; …

comparing $(n-1)$ and (n), we have $p_{n-1} = ap_n$.

Hence $p_1 = ap_2 = a^2 p_3 = \cdots = a^{n-1} p_n$, i.e. $p_1 = a^{n-1} p_n$, $p_2 = a^{n-2} p_n$, $\cdots$, $p_{n-1} = ap_n$.

Since $\sum_{i=1}^{n} p_i = 1$, i.e. $p_n \cdot (a^{n-1} + a^{n-2} + \cdots + a + 1) = 1 \Rightarrow p_n = \frac{a-1}{a^n - 1}$

So $\vec{p} = (\frac{a^{n-1}(a-1)}{a^n-1}, \frac{a^{n-2}(a-1)}{a^n-1}, \cdots, \frac{a(a-1)}{a^n-1}, \frac{a-1}{a^n-1})$.

By ①, $v = (a+1)p_1 + p_2 + p_3 + \cdots + p_n = \sum_{i=1}^{n} p_i + ap_1 = 1 + a \cdot \frac{a^{n-1}(a-1)}{a^n-1} = \frac{a^n - 1 + a^{n+1} - a^n}{a^n-1} = \frac{a^{n+1}-1}{a^n-1}$.

Similarly, by principle of indifference, we can get the minimax strategy is

$\vec{q} = (\frac{a-1}{a^n-1}, \frac{a(a-1)}{a^n-1}, \cdots, \frac{a^{n-2}(a-1)}{a^n-1}, \frac{a^{n-1}(a-1)}{a^n-1})$.