MATH4250 Game Theory, 2017-2018 Term 2
Mid-term Examination
Time allowed: 90 mins

1. (8 marks) Let $g(x)$ be the Sprague-Grundy function of the subtraction game with subtraction set $S = \{2, 3, 6\}$. The game terminates when there are 0 or 1 chip remaining. The player who makes the last move wins.

   (a) Find $g(6)$, $g(13)$ and $g(34)$.

   (b) Find all winning moves from the position 13 and 34.

   (c) Find the set of P-positions of the game and prove your assertion.

2. (6 marks) Let $\oplus$ denotes the nim-sum.

   (a) Find x if $x \oplus 10 = 25 \oplus 29$.

   (b) Find all winning moves of the game of nim from the position $(10, 25, 29)$.

3. (8 marks) Consider the following 3 games.
   Game 1: Less than half (In each turn, a player may remove less than half of the chips.)
   Game 2: Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6, 7\}$
   Game 3: 1-pile nim
   Let g_1, g_2, g_3 be the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the three games and g be the Sprague-Grundy function of G.

   (a) Write down the set of P-positions of Game 1. No proof is required.

   (b) Find $g(13, 12, 7)$.

   (c) Find all winning moves of G from the position $(13, 12, 7)$.

4. (8 marks) Let
$$A = \begin{pmatrix} 6 & 9 & 4 & 8 & 3 \\ 5 & 3 & 7 & 6 & 2 \\ 4 & 1 & 6 & 3 & 5 \end{pmatrix}$$

   (a) Write down the reduced matrix obtained by deleting all dominated rows and columns of A.

   (b) Use the reduced matrix to solve the two-person zero sum game with game matrix A, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

5. (8 marks) Use simplex method to solve the game with the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

$$\begin{pmatrix} 3 & -1 & -1 \\ 2 & 0 & -2 \\ -3 & 1 & 3 \end{pmatrix}$$

6. (12 marks) Let A be an $n \times n$ square matrix and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

   (a) Prove the following statements.

      (i) If A is a symmetric matrix, that is $A^T = A$, and there exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T = v\mathbf{1}^T$ where $v \in \mathbb{R}$ is a real number, then v is the value of A.

      (ii) There exists a square matrix A, a probability vector $\mathbf{y}$ and a real number v such that $A\mathbf{y}^T = v\mathbf{1}^T$ but v is not the value of A.

   (b) Suppose

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

      (i) Find a vector $\mathbf{x} = (1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ and a real number a such that

$$A\mathbf{x}^T = (0, 0, 0, 0, a)^T$$

      (ii) Find a vector $\mathbf{y} = (1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$ and a real number b such that

$$A\mathbf{y}^T = (1, 1, 1, 1, b)^T$$

      (iii) Find the maximin strategy, the minimax strategy and the value of A. (Hint: Find real numbers $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{q} = \alpha\mathbf{x} + \beta\mathbf{y}$ satisfies $A\mathbf{q}^T = v\mathbf{1}^T$ for some $v \in \mathbb{R}$.)

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