1(a) $x \oplus 13 \oplus 23 = 28$

$\Rightarrow x = 28 \oplus 23 \oplus 13$

$\Rightarrow x = 6$

$$\begin{array}{r} 11100 \\ 10111 \\ +\ 1101 \\ \hline 110_2 \end{array}$$

b) From (a), we know that $(13, 23, 28)$ is not a P-position.

Since $13 \oplus 23 = 11010_2 = 26 < 28$

$23 \oplus 28 = 1011_2 = 11 < 13$

$13 \oplus 28 = 10001_2 = 17 < 23$,

So the winning moves are $(13, 23, 26)$, $(11, 23, 28)$, $(13, 17, 28)$.

2a $g(5,1) = 1$, $g(2,3) = 2$, $g(99,100) = 2$.

b) Since $g(8,5) = 0$, which is a P-position, so there is no winning moves.

c) $P = \{(x,y) \in \mathbb{N} \times \mathbb{N} : x - y \equiv 0 \pmod 3\}$

d) i) $(0,0)$ is the terminal position and $0 - 0 \equiv 0 \pmod 3$.

ii) Take any position $p = (p_1, p_2) \in P$, i.e. $p_1 - p_2 \equiv 0 \pmod 3$. Then p can move to position $q \triangleq (q_1, q_2)$, $q = (p_1 - 1, p_2), (p_1 - 2, p_2), (p_1, p_2 - 1), (p_1, p_2 - 2), (p_1 + 1, p_2 - 1), (p_1 + 2, p_2 - 2)$, where $q_1 - q_2 \equiv 2, 1, 1, 2, 2, 1 \pmod 3$, $\not\equiv 0$, so any move from any position $p \in P$ reaches a position $q \notin P$.

iii) Suppose $q \notin P$. Then $q_1 - q_2 \equiv 1, 2 \pmod 3$. When $q_1 - q_2 \equiv \pmod 3$, then (q_1, q_2) can move to $(q_1 - 1, q_2)$ or $(q_1 + 1, q_2 - 1)$ or $(q_1, q_2 - 2)$, which their difference is 0. When $q_1 - q_2 \equiv 2 \pmod 3$, then (q_1, q_2) can move to $(q_1, q_2 - 1)$, $(q_1 - 2, q_2)$ or $(q_1 + 2, q_2 - 2)$, which their difference is 0.

Thus, $P = \{(x,y) \in \mathbb{N} \times \mathbb{N} ; x - y \equiv 0 \pmod 3\}$.

3a) $g_1(x) = x$, $g_2(x) \equiv x \pmod 7$.

For $g_2(x)$,

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$g_3(x)$	0	1	2	0	1	2	3	4	0	1	2	0	1

We can see that
$$g_3(x) = \begin{cases} 0, & \text{if } x \equiv 0, 3 \pmod 8 \\ 1, & \text{if } x \equiv 1, 4 \pmod 8 \\ 2, & \text{if } x \equiv 2, 5 \pmod 8. \\ 3, & \text{if } x \equiv 6 \pmod 8 \\ 4, & \text{if } x \equiv 7 \pmod 8. \end{cases}$$

So $g_1(7) = 7$, $g_2(19) = 5$, $g_3(15) = 4$.

b) $g(7, 19, 15) = g_1(7) \oplus g_2(19) \oplus g_3(15)$
$= 7 \oplus 5 \oplus 4$
$= 6$.

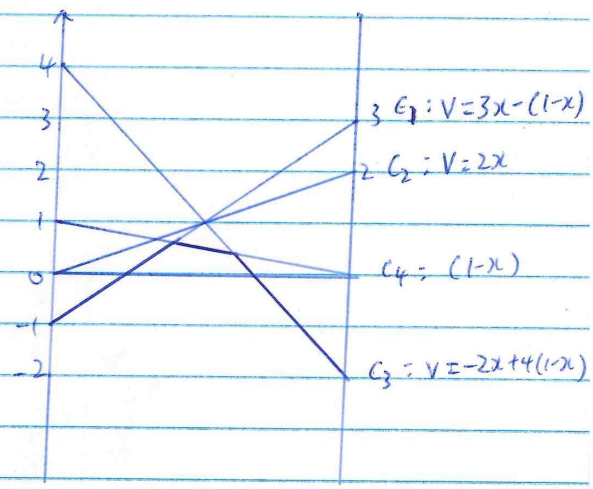
c) Let the winning move of G from the position $(14, 17, 24)$ be (x, y, z).
We can make a move in exactly one of the Game 1, 2, 3 (while the other two remain unchanged) s.t. $g_1(x) = 1_2 = 1$, $g_2(x) = 11_2 = 3$ or $g_3(x) = 10_2 = 2$.
For $g_1(x) = 1$, we can take $x = 1$.
For $g_2(x) = 3$, we can take $x = 17$.
For $g_3(x) = 2$, we can take $x = 13$
So the winning moves are $(1, 19, 15)$, $(7, 17, 15)$, $(7, 19, 13)$.

4a We may delete column 2 as it is dominated by column 5, so we get

$$\begin{pmatrix} 3 & 2 & -2 & 0 \\ 2 & 1 & -3 & -1 \\ -1 & 0 & 4 & 1 \end{pmatrix}$$

We can delete row 2 as it is dominated by row 1, so we get

$$A' = \begin{pmatrix} 3 & 2 & -2 & 0 \\ -1 & 0 & 4 & 1 \end{pmatrix},$$



b) By drawing the lower envelope, the maximum point of it is the intersection point of C_1 and C_4. By solving

C_1: $V = 4x - 1$

C_4: $V = -x + 1$

$x = \frac{2}{5}$, $V = \frac{3}{5}$

For the minimax strategy $\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \end{pmatrix}$

$\Rightarrow \quad y_1 = \frac{1}{5}, \quad y_4 = \frac{4}{5}$

Hence the value of the game is $V = \frac{3}{5}$,

the maximin strategy for row player is $(\frac{2}{5}, 0, \frac{3}{5})$.

the minimax strategy for column player is $(\frac{1}{5}, 0, 0, 0, \frac{4}{5})$.

5 Add $k=2$ to get $\begin{pmatrix} 3 & 1 & 4 \\ 5 & 2 & 0 \\ 1 & 3 & 2 \end{pmatrix}$

Applying simplex method, we have.

	y_1	y_2	y_3	-1
x_1	3	1	4	1
x_2	5^*	2	0	1
x_3	1	3	2	1
-1	1	1	1	0

$\rightarrow$

	x_2	y_2	y_3	-1
x_1	$-\frac{3}{5}$	$-\frac{1}{5}$	4^*	$\frac{2}{5}$
y_1	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{1}{5}$
x_3	$-\frac{1}{5}$	$\frac{13}{5}$	2	$\frac{4}{5}$
-1	$-\frac{1}{5}$	$\frac{3}{5}$	1	$-\frac{1}{5}$

$\rightarrow$

	x_2	y_2	x_1	-1
y_3	$-\frac{3}{20}$	$-\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{10}$
y_1	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{1}{5}$
x_3	$\frac{1}{10}$	$\frac{27}{10}^*$	$-\frac{1}{2}$	$\frac{3}{5}$
-1	$-\frac{1}{20}$	$\frac{13}{20}$	$-\frac{1}{4}$	$-\frac{3}{10}$

$\rightarrow$

	x_2	x_3	x_1	-1
y_3	$-\frac{4}{27}$	$-\frac{1}{54}$	$\frac{13}{54}$	$\frac{1}{9}$
y_1	$\frac{2}{9}$	$-\frac{4}{27}$	$\frac{2}{27}$	$\frac{1}{9}$
y_2	$\frac{1}{27}$	$\frac{10}{27}$	$-\frac{5}{27}$	$\frac{2}{9}$
-1	$-\frac{2}{27}$	$-\frac{13}{54}$	$-\frac{7}{54}$	$-\frac{4}{9}$

The independent variables are $x_4, x_5, x_6, y_4, y_5, y_6$.

The basic solution is $x_4 = x_5 = x_6 = 0$, $x_1 = \frac{7}{54}$, $x_2 = \frac{2}{27}$, $\frac{13}{54}$.

$y_4 = y_5 = y_6 = 0$, $y_1 = \frac{1}{9}$, $y_2 = \frac{2}{9}$, $y_3 = \frac{1}{9}$.

The optimal value is $d = \frac{4}{9}$,

The maximin strategy is $p = \frac{9}{4}\left(\frac{7}{54}, \frac{2}{27}, \frac{13}{54}\right) = \left(\frac{7}{24}, \frac{1}{6}, \frac{13}{24}\right)$.

$q = \frac{9}{4}\left(\frac{1}{9}, \frac{2}{9}, \frac{1}{9}\right) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

The value of the game $= \frac{9}{4} - 2 = \frac{1}{4}$.

6a) $Ay^T = 0$

$$\begin{pmatrix} a_1 & -a_1 & 0 & \cdots & 0 \\ 0 & a_2 & -a_2 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ -a_n & 0 & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$a_1 y_1 - a_1 y_2 = 0$$
$$a_2 y_2 - a_2 y_3 = 0$$
$$\vdots$$
$$a_n y_n - a_n y_1 = 0$$

Since $a_1, \cdots, a_n > 0$, So $y_1 = y_2 = \cdots = y_n$.

Also y is probability vector, $\sum_{i=1}^n y_i = 1$, hence $y_1 = \cdots = y_n = \frac{1}{n}$.

6c) let $B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, then

$$By^T = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

But

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{c} \text{min} \\ -1 \\ -1 \\ -1 \end{array} \\ \text{max} \quad 1 \quad 0 \quad -1 & \end{array}$$

So maximin and minimax is -1, hence value $= -1 \neq 0$. The statement is wrong.

b) Assume II has optimal strategy, by principle of indifference, I's optimal strategy p satisfies $\sum_{i=1}^n p_i a_{ij} = V$, $j = 1, \cdots, n$.

$\Rightarrow$

$$p_1 a_1 - p_n a_n = V$$
$$-p_1 a_1 + p_2 a_2 = V$$
$$-p_2 a_2 + p_3 a_3 = V$$
$$\vdots$$
$$-p_{n-1} a_{n-1} + p_n a_n = V$$

$\Rightarrow$ $nV = 0$, $V = 0$.

$\Rightarrow$ $\frac{a_1}{a_i} = \frac{p_i}{p_1}$, $p_i = \frac{a_1}{a_i} p_1$

$\Rightarrow 1 = p_1 + p_2 + \cdots + p_n = p_1 + \frac{a_1}{a_2} p_1 + \cdots + \frac{a_1}{a_n} p_1 = \left(1 + \frac{a_1}{a_2} + \cdots + \frac{a_1}{a_n}\right) p_1$

$\Rightarrow$ $p_1 = \frac{1}{a_1\left(\sum_{i=1}^n \frac{1}{a_i}\right)}$, $\Rightarrow p_j = \frac{1}{a_j\left(\sum_{i=1}^n \frac{1}{a_i}\right)}$