Real Analysis Midterm Questions

1. State without proof:

   (a) Vitali Cover Lemma;

   (b) Differentiation Theorem for monotone increasing functions and the corresponding result for functions of bounded variations.

2. Let $f \in L_1[a,b]$. Show that $\frac{d}{dx}\int_a^x f = f$ a.e on $[a,b]$ progressively for the cases:

   (a) f is bounded;

   (b) $f \geq 0$ a.e on $[a,b]$.

3. Let $f \in ABC[a,b]$ be such that $m(E) = b - a$ were $E := \{x \in (a,b) : f'(x) = 0\}$. Show that

   (a) $\forall \epsilon > 0$, $\exists$ a Vitali cover $\mathcal{C}$ of E such that the slope of f for each interval in $\mathcal{C}$ is bounded by ϵ:
   $$\left|\frac{f(d) - f(c)}{c - d}\right| < \epsilon, \quad [c,d] \in \mathcal{C}.$$

   (b) f is a constant function on $[a,b]$.

4. Which of the following statements are correct? Prove each of the correct statements. For each of the incorrect statements, provide a counter-example and modify the statement to make it a good one.

   (a) For any subset $E \subset \mathbb{R}$, E is measurable iff $\forall \epsilon > 0$, there exist open G and closed F with $F \subset E \subset G$ such that $m(G \setminus F) < \epsilon$.

   (b) Let $\{f_n\}$ be a sequence of integrable functions on $E \in \mathcal{M}$ which is convergent pointwisely to an integrable function f on E. Then
   $$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

   (c) Let $\{f_n\}$ be a monotonically decreasing sequence of integrable functions on $E \in \mathcal{M}$ which converges to the zero function a.e on E. Then
   $$\lim_{n\to\infty} \int_E f_n = 0$$

Remark: Regarding Q2, Q3 and originally correct statements in Q4, if they are theorem(s) or part(s) of theorem(s), you are allowed to make use of earlier results (in the sequence of our course materials) but not the theorem itself. Regarding each of the originally wrong statements in Q4, you are not required to supply full proof for the revised statement. It suffices to quote a theorem learnt in the course.