MATH 4050 Real Analysis

Tutorial 9 (March 22, 24)

The following were discussed in the tutorial this week.

1. Recall the definition of absolutely continuous function on $\mathbb{R}$.

2. Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then f maps sets of measure zero to sets of measure zero. Give counter-examples to show that the converse does not hold in general.

3. State the fact that a function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if and only if the following three conditions are all satisfied:

   (a) f is continuous;

   (b) f is of bounded variation;

   (c) f maps sets of measure zero to sets of measure zero.

4. Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then f maps measurable sets to measurable sets.

5. Recall the Vitali lemma. Apply the Vitali lemma to prove the Lebesgue density theorem:

   Let $E \subseteq \mathbb{R}$ be measurable. Then

   $$\lim_{r \to 0^+} \frac{m(E \cap [x-r, x+r])}{m([x-r, x+r])} = \chi_E(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

6. Apply the Lebesgue density theorem to prove the Steinhauss theorem:

   Suppose $E \subseteq \mathbb{R}$ is measurable with $m(E) > 0$. Then the difference set

   $$E - E := \{x - y : x, y \in E\}$$

   contains an interval $(-h, h)$ for some $h > 0$.

   More generally, if $A, B \subseteq \mathbb{R}$ are measurable with $m(A), m(B) > 0$, then $A + B$ contains an interval.