MATH 4050 Real Analysis

Tutorial 12 (April 12)

The following were discussed in the tutorial this week.

Let E be a measurable subset of $\mathbb{R}$.

1. Let $1 \leq p < \infty$. Recall that

$$L^p(E) = \{f \text{ measurable function on } E \text{ and } \|f\|_p := \left(\int_E |f|^p\right)^{1/p} < \infty\},$$

where we identify functions that are equal almost everywhere on E. The Riesz-Fisher theorem asserts that $(L^p(E), \|\cdot\|_p)$ is a Banach space.

2. Let

$$L^\infty(E) = \{\text{bounded measurable functions on } E\},$$

where we again identify functions that are equal almost everywhere on E. Define

$$\|f\|_\infty = \inf\{\alpha : m(\{x \in E : |f(x)| > \alpha\}) = 0\}.$$

Then it is easy to see that

$$\|f\|_\infty \leq \lambda \quad \Longleftrightarrow \quad |f(x)| \leq \lambda \text{ for a.e. } x \in E.$$

Moreover $(L^\infty(E), \|\cdot\|_\infty)$ is also a Banach space.

3. Recall the Hölder inequality: if $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q.$$

4. Suppose $m(E) < \infty$. Show that if $1 \leq p < q \leq \infty$, then $L^q(E) \subseteq L^p(E)$. The assumption "$m(E) < \infty$" cannot be dropped.

5. Show that if $1 \leq r < p < s \leq \infty$, then $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$. In particular, $L^r(E) \cap L^s(E) \subseteq L^p(E)$.

6. Suppose $\|f\|_r < \infty$ for some $1 \leq r < \infty$. Show that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$