MATH 4050 Real Analysis

Tutorial 10 (March 29, 31)

Let $[a, b] \subseteq \mathbb{R}$ be a non-degenerate closed bounded interval, and $f : [a, b] \to \mathbb{R}$ be a measurable functions.

The following were discussed in the tutorial this week.

1. For $\alpha > 0$, let
$$E_\alpha = \{x \in [a, b] : f'(x) \text{ exists and } |f'(x)| < \alpha\}.$$
Show that $m^*(f(E_\alpha)) \leq \alpha m^*(E_\alpha)$.

   Remark: In the proof, we define an increasing sequence of subsets of E_α such that $E = \bigcup_n E_n$. As E_n may not be measurable, it is not clear whether $m^*(E) = \lim_{n\to\infty} m^*(E_n)$. Nevertheless, by the outer regularity of m^*, one can indeed prove this limit.

2. Let $E \subseteq [a, b]$ be measurable. Suppose $f'(x)$ exists for all $x \in E$. Show that $m^*(f(E)) \leq \int_E |f'|$.

3. Apply the result in 2 to prove the Banach-Zarecki theorem: $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if the following conditions are all satisfied.
   (a) f is continuous.
   (b) f is of bounded variation.
   (c) f has the Luzin N property: f maps sets of measure zero to sets of measure zero.

4. Recall the Jordan decomposition for functions of bounded variation: $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there is a pair of increasing functions g, h on $[a, b]$ such that $f = g - h$ and
$$g(b) - g(a) + h(b) - h(a) = T_a^b(f).$$
The pair (g, h) is unique up to a constant: if $(g_1, h_1), (g_2, h_2)$ are two such pairs, then there is a constant c such that
$$g_1 - g_2 = h_1 - h_2 = c.$$

5. Let f be of bounded variation on $[a, b]$. Show that
$$\int_a^b |f'| \leq T_a^b(f) \quad \text{and} \quad \int_a^b (f')^+ \leq P_a^b(f).$$
Furthermore, show that f is absolutely continuous on $[a, b]$ if and only if
$$\int_a^b |f'| = T_a^b(f).$$