Solution 3

1. Show that $\inf X \geq \inf Y$ whenever $X \subseteq Y (\subseteq \mathbb{R})$ and hence that $m^*(A) \uparrow$ (i.e. $m^*(A) \leq m^*(B)$ if $A \subseteq B (\subseteq \mathbb{R})$).

   Solution. Let $x \in X$. Then $x \in Y$, and hence by the definition of infimum, $x \geq \inf Y$. Since $x \in X$ is arbitrary, we have $\inf X \geq \inf Y$. The last statement follows immediately from the definition

   $$m^*(A) := \inf\{\sum_{k=1}^{\infty} \ell(I_k) : \{I_k\}_{k=1}^{\infty} \text{ is a countable open-interval cover of } A\},$$

   and the fact that if $A \subseteq B \subseteq \mathbb{R}$, then any countable interval cover of B is also a countable interval cover of A. ◀

2. Let $\mathcal{A}$ be an algebra of subsets of X. Show that $\mathcal{A}$ is a σ-algebra if (and only if) $\mathcal{A}$ is stable with respect to countable disjoint unions:

   $$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A} \;\text{ whenever } A_n \in \mathcal{A}\; \forall\, n \in \mathbb{N} \text{ and } A_m \cap A_n = \emptyset\; \forall\, m \neq n.$$

   Solution. Suppose $\mathcal{A}$ is an algebra of subset of X that is stable with respect to countable disjoint unions. To show that $\mathcal{A}$ is a σ-algebra, it suffices to show that $\mathcal{A}$ is stable with respect to countable (but not necessarily disjoint) union. Let $B_n \in \mathcal{A}$ for $n \in \mathbb{N}$. Define

   $$C_1 := B_1 \quad \text{and} \quad C_n := B_n \setminus \bigcup_{k=1}^{n-1} B_k \;\text{ for } n \geq 2.$$

   Clearly the collection $\{C_n\}_{n=1}^{\infty}$ is pairwise disjoint, and each $C_n \in \mathcal{A}$ since $\mathcal{A}$ is an algebra. Moreover,

   $$\begin{aligned} C_1 \cup C_2 &= B_1 \cup (B_2 \setminus B_1) = B_1 \cup B_2, \\ C_1 \cup C_2 \cup C_3 &= B_1 \cup B_2 \cup (B_3 \setminus (B_1 \cup B_2)) = B_1 \cup B_2 \cup B_3, \\ &\vdots \end{aligned}$$

   and so on. Hence $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n \in \mathcal{A}$. ◀

3. Suppose $[a, b]\,(\subseteq \mathbb{R})$ is covered by a finite family $\mathcal{C}$ of open intervals. Show that $b - a \leq$ sum of lengths of intervals in $\mathcal{C}$ (by MI to $n := \#(\mathcal{C})$, the number of elements of $\mathcal{C}$).

   Solution. Let $P(n)$ be the statement: if $[a, b]$ is a closed bounded interval that is covered by a finite family $\mathcal{C}$ of open intervals with $\#(\mathcal{C}) = n$, then $b - a \leq$ sum of lengths of intervals in $\mathcal{C}$.

   Suppose $\#(\mathcal{C}) = 1$ and $\mathcal{C} = \{[c, d]\}$. Then clearly $b - a \leq d - c$. Hence $P(1)$ is true.

   Assume that $P(k)$ is true. Suppose $[a, b]$ is a closed bounded interval that is covered by a finite family $\mathcal{C} = \{(c_i, d_i)\}_{i=1}^{k+1}$ of open intervals. Without loss of generality, we may assume that $a \in (c_1, d_1)$. Then $[d_1, b]$ is a closed bounded interval covered by $\{(c_i, d_i)\}_{i=2}^{k+1}$. Now the induction assumption implies that

   $$b - d_1 \leq \sum_{i=2}^{k+1} |c_i - d_i|,$$

and hence

$$b-a=(d_1-a)+(b-d_1)\le |c_1-d_1|+\sum_{i=2}^{k+1}|c_i-d_i|=\sum_{i=1}^{k+1}|c_i-d_i|.$$

So $P(k+1)$ is true.

By MI, $P(n)$ is true for all $n\in\mathbb{N}$. ◀

4. (cf. Royden 3rd, p.52, Q51) Upper/Lower Envelopes of $f:[a,b]\to\mathbb{R}$.

Define $h,g:[a,b]\to[-\infty,\infty]$ by

$$h(y):=\inf\{h_\delta(y):\delta>0\}\quad\text{for all } y\in[a,b],$$

where $h_\delta(y):=\sup\{f(x):x\in[a,b],|x-y|<\delta\}$; and

$$g(y):=\sup\{g_\delta(y):\delta>0\}\quad\text{for all } y\in[a,b],$$

where $g_\delta(y):=\inf\{f(x):x\in[a,b],|x-y|<\delta\}$. Prove the following:

(a) $g\le f\le h$ pointwisely on $[a,b]$, and for all $x\in[a,b]$, $g(x)=f(x)$ if and only if f is lower semicontinuous (l.s.c) at x ($f(x)=h(x)$ if and only if f is upper semicontinuous (u.s.c) at x), so $g(x)=h(x)$ if and only if f is continuous at x.

(b) If f is bounded (so g,h are real-valued), then g is l.s.c and h is u.s.c.

(c) If ϕ is a l.s.c function on $[a,b]$ such that $\phi\le f$ (pointwise) on $[a,b]$, then $\phi\le g$. State and show the corresponding result for h.

(d) Let $C_n:=\{x\in[a,b]:h(x)-g(x)<\frac{1}{n}\}$ for all $n\in\mathbb{N}$. Then $C:=\bigcap_{n=1}^{\infty}C_n$ is exactly the set of all continuity points of f and is a G_δ-set.

Note: More suggestive notations for g,h are $\underline{f},\overline{f}$.

Solution. (a) Clearly $g_\delta(x)\le f(x)\le h_\delta(x)$ for all $x\in[a,b]$ and $\delta>0$. Hence $g\le f\le h$ pointwisely on $[a,b]$.

Suppose f is l.s.c at x, that is, for all $\varepsilon>0$, there exists $\delta>0$ such that $f(x)-\varepsilon<f(y)$ whenever $y\in[a,b]$ and $|y-x|<\delta$. Then $f(x)-\varepsilon\le g_\delta(x)\le g(x)$. Since $\varepsilon>0$ is arbitrary, we have $f(x)\le g(x)$, and hence $f(x)=g(x)$.

On the other hand, suppose $f(x)=g(x)$. Let $\varepsilon>0$. Fix $\delta>0$ such that $g(x)<g_\delta(x)+\varepsilon$. Since $(y-\delta/2,y+\delta/2)\subseteq(x-\delta,x+\delta)$ whenever $y\in(x-\delta/2,x+\delta/2)\cap[a,b]$, then it follows from the definition that

$$g_\delta(x)\le g_{\delta/2}(y)\le g(y),$$

and hence $f(x)-\varepsilon=g(x)-\varepsilon<g_\delta(x)\le g(y)\le f(y)$. Therefore f is l.s.c at x.

Similarly, one can show that $f(x)=h(x)$ if and only if f is u.s.c at x.

The last assertion now follows immediately from above and the simple fact that f is continuous at x if and only if it is both l.s.c and u.s.c at x.

(b) The proof is essentially the same as that in the second part of (a). Let $x\in[a,b]$ and $\varepsilon>0$. Since g is real-valued, we can find $\delta>0$ such that $g(x)<g_\delta(x)+\varepsilon$. Note that $(y-\delta/2,y+\delta/2)\subseteq(x-\delta,x+\delta)$ if $|x-y|<\delta/2$. It follows from the definition that whenever $y\in(x-\delta/2,x+\delta/2)\cap[a,b]$, we have

$$g(x)-\varepsilon<g_\delta(x)\le g_{\delta/2}(y)\le g(y).$$

Therefore g is l.s.c on $[a,b]$.

Similarly one can show that h is u.s.c on $[a,b]$.

(c) It suffices to prove that if ϕ is l.s.c at x and $\phi \le f$ on $[a, b]$, then $\phi(x) \le \underline{f}(x)$. From the definition,

$$\underline{\phi}(x) := \sup_{\delta>0} (\inf\{\phi(y) : y \in [a, b] : |x - y| < \delta\}) \le g(x)$$

Since ϕ is l.s.c at x, we have $\underline{\phi}(x) = \phi(x)$ by (a), and the result follows.
Similarly, one can prove the corresponding result for h: if ψ is a u.s.c function on $[a, b]$ such that $f \le \psi$ on $[a, b]$, then $h \le \psi$.

(d) By (a), we have

$$\begin{aligned}\{x \in [a, b] : f \text{ continuous at } x\} &= \{x \in [a, b] : g(x) = h(x)\} \\ &= \bigcap_{n=1}^{\infty} \{x \in [a, b] : h(x) - g(x) < 1/n\} \\ &= \bigcap_{n=1}^{\infty} C_n = C.\end{aligned}$$

To see that C is a G_δ-set (in $[a, b]$), it suffices to show that, given any $\lambda > 0$, $A := \{x \in [a, b] : h(x) - g(x) < \lambda\}$ is open in $[a, b]$. Let $x_0 \in A$. Then there exists $\gamma \in (0, 1)$ such that $h(x_0) - g(x_0) < \gamma\lambda$. By the definitions of h, g, there exists $\delta_1, \delta_2 > 0$ such that $h_{\delta_1}(x_0) - g_{\delta_2}(x_0) < \gamma\lambda$, and hence

$$f(y) - f(z) < \gamma\lambda \quad \text{whenever } y, z \in [a, b] \text{ and } |y - x_0| < \delta_1, |z - x_0| < \delta_2.$$

In particular, if $x \in [a, b]$ and $|x - x_0| < \delta := \min\{\delta_1, \delta_2\}/2$, then

$$f(y) - f(z) < \gamma\lambda \quad \text{whenever } y, z \in [a, b] \text{ and } |y - x|, |z - x| < \delta.$$

Thus $h_\delta(x) - g_\delta(x) \le \gamma\lambda$, so that $h(x) - g(x) \le \gamma\lambda < \lambda$ whenever $x \in [a, b]$ and $|x - x_0| < \delta$. Therefore A is an open subset of $[a, b]$.

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5. Let $f : [a, b] \to [m, M]$. For each $P \in \text{Par}[a, b]$, let $u(f; P)$ and $U(f; P)$ denote the lower/upper Riemann-sum functions. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}\ \forall n$ and $\|P_n\| \to 0$ ($\|P\|$ is the max subinterval length of P). Show that, $\forall\, x \in [a, b] \setminus A$

$$\lim_n (u(f; P_n))(x) = \underline{f}(x) \quad \text{and} \quad \lim_n (U(f; P_n))(x) = \overline{f}(x),$$

where A denotes the union of all end-points of $P_n\ \forall n$.

Solution. Let ϕ, ψ be bounded functions on $[a, b]$, and P, Q be partitions on $[a, b]$. It is clear from the definitions that the lower and upper Riemann-sum functions satisfy the following properties

(i) $u(\phi; P) \le \phi \le U(\phi; P)$.
(ii) $u(\phi; P) \le u(\phi; Q)$ and $U(\phi; Q) \le U(\phi; P)$ if $P \subseteq Q$.
(iii) $u(\phi; P) \le u(\psi; P)$ and $U(\phi; P) \le U(\psi; P)$ if $\phi \le \psi$.
(iv) $u(\phi; P)$ and $U(\phi; P)$ are continuous except at the end-points of P.

Let $\{P_n\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}\ \forall n$ and $\|P_n\| \to 0$. Then (ii) implies that $u(f; P_n)$ is an increasing sequence of functions, so that $\lim_n u(f; P_n)$ exists. Moreover we have

$$u(\underline{f}; P_n)(x) \le u(f; P_n)(x) \le \underline{f}(x), \quad \text{for all } x \in [a,b] \setminus A, \tag{1}$$

where the first inequality follows from 4(a) and (iii), while the second one follows from (the proof of) 4(c), (i) and (iv).

Fix $x \in [a,b] \setminus A$. Since $\underline{f}$ is l.s.c at x, there exists $\delta > 0$ such that

$$\underline{f}(x) - \varepsilon < \underline{f}(y) \quad \text{whenever } y \in [a,b] \text{ and } |y - x| < \delta. \tag{2}$$

Choose N so large such that $\|P_N\| < \delta$. Suppose $a = a_0 < a_1 < \cdots < a_k = b$ are the end-points of P_N. Then (2) implies that

$$\underline{f}(x) - \varepsilon \le \sum_{i=1}^{k} \Big(\inf_{y \in (x_{i-1}, x_i)} \underline{f}(y) \Big) \chi_{(x_{i-1}, x_i)}(x) = u(\underline{f}; P_N)(x).$$

Combining this with (1) and (ii), we have

$$\underline{f}(x) - \varepsilon \le u(f; P_N)(x) \le u(f; P_n)(x) \le \underline{f}(x) \quad \text{for } n \ge N,$$

and hence $\lim_n u(f; P_n)(x) = \underline{f}(x)$.

Similarly we can show that $\lim_n U(f; P_n)(x) = \overline{f}(x)$ for $x \in [a,b] \setminus A$.

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