Solution 2

In this assignment, $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers. E is a subset of $\mathbb{R}$.

Recall that the limit superior of $\{x_n\}$ is defined by

$$\limsup x_n := \inf_n \sup_{k \geq n} x_k.$$

Clearly $z_n := \sup_{k \geq n} x_k$ is monotone decreasing, and hence

$$\lim_n z_n = \inf_n z_n = \limsup x_n, \tag{1}$$

where the limit is taken in the extended real number. Similarly the limit inferior of $\{x_n\}$ is given by

$$\liminf x_n := \sup_n \inf_{k \geq n} x_k = \lim_n \inf_{k \geq n} x_k. \tag{2}$$

1.* (3rd: P.39, Q12)

Show that $x = \lim x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence that converges to x. How about $x \in \{-\infty, \infty\}$?

Solution. ($\Longrightarrow$) Suppose $\lim x_n = x$. Then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x. Therefore $\{x_{n_k}\}$ has itself as a further subsequence that converges to x.

($\Longleftarrow$) Suppose on the contrary that $\{x_n\}$ does not converge to x. Then there exists $\varepsilon_0 > 0$ such that for all $N \in \mathbb{N}$, there is $n > N$ such that

$$|x_n - x| \geq \varepsilon_0.$$

Take $N = 1$, then we can find $n_1 > 1$ such that $|x_{n_1} - x| \geq \varepsilon_0$. Take $N = n_1$, we can find $n_2 > n_1$ such that $|x_{n_2} - x| \geq \varepsilon_0$. Continue in this way, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \text{for } k \in \mathbb{N}.$$

Now $\{x_{n_k}\}$ has no further subsequence that converges to x.

Similar results hold if $x = -\infty$ or ∞. ◀

2. (3rd: P.39, Q13)

Show that the real number l is the limit superior of the sequence $\{x_n\}$ if and only if (i) given $\varepsilon > 0$, $\exists\, n$ such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) given $\varepsilon > 0$ and n, $\exists\, k \geq n$ such that $x_k > l - \varepsilon$.

Solution. We show that (1) $\limsup x_n < l'$ if and only if $\exists\, n$ such that $x_k < l'$ for all $k \geq n$; and (2) $\limsup x_n > l''$ if and only if for all n, $\exists\, k \geq n$ such that $x_k > l''$.

(1): By the definition of supremum and infinmum,

$$\begin{aligned}\limsup x_n < l' &\iff \inf_n \sup_{k \geq n} x_k < l' \iff (\exists\, n)(\sup_{k \geq n} x_k < l') \\ &\iff (\exists\, n)(\forall\, k \geq n)(x_k < l').\end{aligned}$$

(2): By the definition of supremum and infinmum,

$$\begin{aligned}\limsup x_n > l'' &\iff \inf_n \sup_{k \geq n} x_k > l'' \iff (\forall\, n)(\sup_{k \geq n} x_k > l'') \\ &\iff (\forall\, n)(\exists\, k \geq n)(x_k > l'').\end{aligned}$$

Remark:

(a) Similar results hold for limit inferior.

(b) (1), (2) may fail if "$<$" ("$>$") is replaced by "$\leq$" ("$\geq$").

Now the desired statement follows immediately once we note that $\limsup x_n = l$ if and only if given any $\varepsilon > 0$, $l - \varepsilon < \limsup x_n < l + \varepsilon$.

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3.* (3rd: P.39, Q14)

Show that $\limsup x_n = \infty$ if and only if given Δ and n, $\exists\, k \geq n$ such that $x_k > \Delta$.

Solution. The statement follows immediately from (2) in question 2 and the fact that $x = \infty$ if and only if $x > \Delta$ for any $\Delta \in \mathbb{R}$. ◀

4. (3rd: P.39, Q15)

Show that $\liminf x_n \leq \limsup x_n$ and $\liminf x_n = \limsup x_n = l$ if and only if $l = \lim x_n$.

Solution. Clearly $\inf_{k\geq n} x_k \leq \sup_{k\geq n} x_k$ for all $n \geq 1$. Hence, by (1) and (2), and letting $n \to \infty$, we have

$$\liminf x_n = \lim_n \inf_{k\geq n} x_k \leq \lim_n \sup_{k\geq n} x_k = \limsup x_n.$$

Suppose $\liminf x_n = \limsup x_n = l$. Then by the results in question 2, given any $\varepsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$x_j > l - \varepsilon \quad \text{for all } j \geq n_1$$

and

$$x_k < l + \varepsilon \quad \text{for all } k \geq n_2.$$

Hence $l - \varepsilon < x_n < l + \varepsilon$ for all $n \geq \max\{n_1, n_2\}$. Thus we have $\lim x_n = l$. The converse can be proved by reversing the argument above.

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5.* (3rd: P.39, Q16)

Prove that

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

provided the right and left sides are not of the form $\infty - \infty$.

Solution. For all $n \geq 1$,

$$x_k + \inf_{j\geq n} y_j \leq x_k + y_k \quad \text{whenever } k \geq n,$$

so that

$$\sup_{k\geq n} x_k + \inf_{j\geq n} y_j \leq \sup_{k\geq n}(x_k + y_k).$$

By (1) and (2), we can let $n \to \infty$ on both sides and obtain

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n),$$

provided the left side is not of the form $\infty - \infty$.

On the other hand, for all $n \geq 1$,

$$\sup_{k \geq n}(x_k + y_k) \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k.$$

Again, using (1) and (2), and letting $n \to \infty$, we obtain

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

provided the right side is not of the form $\infty - \infty$.

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6. (3rd: P.39, Q17)

   Prove that if $x_n > 0$ and $y_n \geq 0$, then

   $$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n),$$

   provided the product on the right is not of the form $0 \cdot \infty$.

   Solution. For all $n \geq 1$,

   $$0 \leq x_k y_k \leq (\sup_{j \geq n} x_j)(\sup_{j \geq n} y_j) \quad \text{whenever } k \geq n,$$

   since $x_n, y_n \geq 0$, so that

   $$\sup_{k \geq n}(x_k y_k) \leq (\sup_{k \geq n} x_k)(\sup_{k \geq n} y_k).$$

   Using (1) and (2), and letting $n \to \infty$, we have

   $$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n),$$

   provided the right side is not of the form $0 \cdot \infty$. ◀

7. (3rd: P.46, Q27)

   $x \in \mathbb{R}$ is called a *point of closure* of E if each neighbourhood of x intersects E. Show that x is a point of closure of E if and only if there is a sequence $\{y_n\}$ with $y_n \in E$ and $x = \lim y_n$.

   Solution. Suppose x is a point of closure of E. Then the open ball $B(x, 1/n)$, which is centred at x and of radius $1/n$, intersects E for all $n \geq 1$. Pick $y_n \in E \cap B(x, 1/n)$ for each n. Then $\{y_n\}$ is a sequence in E such that $\lim y_n = x$, since $|y_n - x| < 1/n$ for all n.

   On the other hand, suppose $\{y_n\}$ is a sequence in E such that $x = \lim y_n$. Let U be a neighbourhood of x. Then $y_n \to x$ implies that $y_n \in U$ for all sufficiently large n. In particular, $U \cap E \neq \emptyset$. ◀

8. (3rd: P.46, Q28; 4th: P.20, Q30(i))

   A number x is called an *accumulation point* of a set E if it is a point of closure of $E \setminus \{x\}$. Show that the set E' of accumulation points of E is a closed set.

Solution. We would like to show that the complement of E' is open. Let $x \in (E')^c$. Then x is not a point of closure of $E \setminus \{x\}$. Hence, by definition, there is an open neighbourhood U of x such that $U \cap (E \setminus \{x\}) = \emptyset$. We claim that every $y \in U$ is not an accumulation point of E, so that $x \in U \subseteq (E')^c$, and hence $(E')^c$ is open.

Let $y \in U \setminus \{x\}$. Since $U \setminus \{x\}$ is open, there is a neighbourhood V of y such that $V \subseteq U \setminus \{x\}$. Hence

$$V \cap (E \setminus \{y\}) \subseteq (U \setminus \{x\}) \cap E = \emptyset.$$

Thus y is not a point of closure of $E \setminus \{y\}$, that is, y is not an accumulation point of E.

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9. (3rd: P.46, Q29; 4th: P.20, Q30(ii))

Show that $\overline{E} = E \cup E'$.

Solution. Recall that $\overline{E}$ is the set of all point of closure of E. From the definitions, it is clear that $E \cup E' \subseteq \overline{E}$. On the other hand, if $x \in \overline{E} \setminus E$, then for every neighbourhood U of x,

$$U \cap (E \setminus \{x\}) = U \cap E \neq \emptyset.$$

Hence $x \in E'$. Therefore $\overline{E} \subseteq E \cup E'$. ◀

10. (3rd: P.46, Q30; 4th: P.20, Q31)

A set E is called *isolated* if $E \cap E' = \emptyset$. Show that every isolated set of real numbers is countable.

Solution. Suppose E is isolated. Then no point in E is an accumulation point of E, that is, for all $x \in E$, there is $r > 0$ such that $B(x, r) \cap (E \setminus \{x\}) = \emptyset$. Hence

$$E = \bigcup_{n,k} \{x \in E \cap [-k, k] : B(x, 1/n) \cap (E \setminus \{x\}) = \emptyset\} =: \bigcup_{n,k} E_{n,k}.$$

We will show that $E_{n,k}$ is a finite set for each $n, k \geq 1$. Then E is countable since it is a countable union of finite sets.

From the definition of $E_{n,k}$, it is clear that

$$B(x, 1/2n) \subseteq [-k-1, k+1] \quad \text{for all } x \in E_{n,k},$$

and

$$B(x, 1/2n) \cap B(y, 1/2n) = \emptyset \quad \text{for all } x, y \in E_{n,k},\ x \neq y,$$

for otherwise, $x \in B(y, 1/n) \cap E_{n,k} \subseteq B(y, 1/n) \cap E \setminus \{y\}$. Hence $E_{n,k}$ is finite, otherwise $\bigcup_{x \in E_{n,k}} B(x, 1/2n)$ is unbounded.

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