1/3 tutorial materials

Example 0.1. *If $f : [0,1] \to \mathbb{R}$ is non-decreasing, then f is measurable. (There is some obvious mistakes in the original argument.)*

Proof. Assume f is strictly increasing first. Let $c \in \mathbb{R}$, if $\exists p \in [0,1]$ such that $f(p) = c$, then

$$f^{-1}[(c, +\infty)] = (p, 1]$$

which is clearly measurable. If not, then we have $f(x) \neq c$ for any $x \in [0,1]$ and

1. $f(0) > c$, or
2. $f(1) < c$, or
3. $f(0) < c < f(1)$.

As f is strictly increasing, the pre-image of the first two cases is either empty set or $[0,1]$. So it suffices to consider the third case. Define

$$a = \sup\{x \in [0,1] : f(x) < c\}.$$

Then

$$\lim_{x \to a^-} f(x) \leq c$$

and for any $x > a$, $f(x) > c$. Hence,

$$f^{-1}[(c, +\infty)] = (a, 1] \text{ or } [a, 1].$$

In all cases, the sub-level set is measurable.

If f is not strictly monotone, then consider $f_n = f + x/n$. f_n is measurable and hence the limit function $\lim_n f_n = f$ is measurable. □

Remark: You can also use the fact that monotone function has countable jumping discontinuity to show the measurability.