Lemma (on Two Runners) Let $f_1, f_2 \in \mathcal{L}_1[a,b]$ s.t. $\int_a^x f_1 = \int_a^x f_2 \ \forall x \in [a,b]$. Then $f_1 = f_2$ a.e. on $[a,b]$.

Proof. Let $f := f_1 - f_2 \in \mathcal{L}_1[a,b]$. Then $\int_a^x f = 0 \ \forall x \in [a,b]$. Hence

I. $\int_I f = 0$ whenever $I \subseteq [a,b]$ is an interval (say with ends c, d so $\int_I f = \int_{(c,d]} f = \int_a^d f - \int_a^c f = 0$ as $(c,d] = [a,d] \setminus [a,c]$)

II. Let G be an open set (so $G = \bigcup_{n=1}^{\infty} I_n$ with disjoint intervals by the Structure Th for open sets). Then

$$\int_{G \cap (a,b)} f = \sum_{n=1}^{\infty} \int_{I_n \cap (a,b)} f = 0 \text{ by I.}$$

III. Let F be a closed set. Then

$$\int_{F \cap (a,b)} f = \int_{(a,b)} f - \int_{(a,b) \cap \tilde{F}} f = 0 - 0 \ (\text{by I \& II})$$

IV. Let $E_+ = \{x \in (a,b) : f(x) > 0\}$ and $E_- = \{x : f(x) < 0\}$. Wish to show both are of measure zero. By symmetry (or apply to $-f$ in place of f), need only show that $m(E_+) = 0$. Since

$$m(E_+) = \sup\{m(F) : \text{closed } F \subseteq E_+\},$$

it suffices to show that $m(F) = 0$, where $F \subseteq E_+$ is closed. This follows easily (?) as $0 = \int_F f$ and $0 \leq f$ on F.

Th 1. Let $f \in \mathcal{L}_1[a,b]$ and $F(x) = \int_a^x f \ \forall x$. Then $F' = f$ a.e. on $[a,b]$

Proof. We know that $F \in ABC[a,b]$ and $F' \in \mathcal{L}_1[a,b]$ with $F'(x) \in \mathbb{R}$ a.e x in $[a,b]$; in particular (with convention that $F(x) = F(b) \ \forall x > b$)

$$(1) \quad \lim_{\delta \to 0+} \left(\int_{x_0}^{x_0+\delta} F \Big/ \delta \right) = F(x_0) \quad \forall x_0 \in [a,b] \quad \text{(by the continuity at } x_0\text{)}$$

and, for a.e $x \in (a,b)$,

$$(2) \quad F'(x) = \lim_{n \to \infty} F_n(x) \quad \text{where } F_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{1/n} \quad (n \in \mathbb{N}).$$

The rest of the proof is divided into three steps.

I. Suppose f is bounded: $\exists M > 0$ s.t. $|f(x)| \leq M \ \forall x$. Then

$$|F_n(x)| = \left| \frac{\int_x^{x+\frac{1}{n}} f}{1/n} \right| \leq \frac{\int_x^{x+\frac{1}{n}} |f|}{1/n} \leq M \quad \forall x.$$

By the BCT it follows from (2) that $\forall c \in [a,b]$

$$\int_a^c F'(x)\,dx = \lim_n \int_a^c F_n(x)\,dx = \lim_n \left(n \int_a^c \left(F(x+\tfrac{1}{n}) - F(x)\right) dx \right)$$

$$= \lim_n \left[n \left(\int_{a+1/n}^{c+1/n} F - \int_a^c F \right) \right] \qquad \left(\text{convention: } \begin{array}{l} F(x) = F(a) \ \forall x < a \\ F(x) = F(b) \ \forall x > b \end{array} \right)$$

$$= \lim_n \left[n \int_c^{c+\frac{1}{n}} F - n \int_a^{a+\frac{1}{n}} F \right]$$

$$= F(c) - F(a) \qquad \text{(by (1) above)}$$

$$= \int_a^c f \qquad \left(\text{as } F(a) = 0 = \int_a^c f\right)$$

By our Lemma (on "Two runners") it follows that $F' = f$ a.e., proving the desired result for the present case.

II. Suppose $f \geq 0$ a.e. on $[a,b]$. Then $F \uparrow$ on $[a,b]$ and so, by ...,

$$\int_a^b F' \leq F\Big|_a^b \leq \int_a^b f, \qquad (3)$$

while, on the other hand, the integrand

$$F' = \frac{d}{dx}\int_a^x f = \frac{d}{dx}\left[\int_a^x (f - f\wedge n + f\wedge n)\right] \quad \left(\text{these functions are finite a.e}\right)$$

$$= \frac{d}{dx}\int_a^x (f - f_n) + \frac{d}{dx}\int_a^x f_n \qquad (f_n := f\wedge n)$$

$$\geq \frac{d}{dx}\int_a^x f_n \qquad \left(\text{as } x \mapsto \int_a^x (f - f_n) \text{ is } \uparrow\right)$$

$$= f_n \quad \text{a.e on } [a,b].$$

Passing to the limit, it follows that $F' \geq f$ a.e on $[a,b]$. and so $\int_a^b F' \geq \int_a^b f$ (so equal by (3)). By the converse result for the monotonicity of Lebesgue integrals, we arrive at $F' = f$ a.e on $[a,b]$.

III. General Case: $f \in \mathcal{L}_1[a,b]$. Then, by II,

$$\frac{d}{dx}\int_a^x f^+ = f^+ \quad \& \quad \frac{d}{dx}\int_a^x f^- = f^- \quad \text{a.e on } [a,b]$$

and so, a.e on $[a,b]$ one has

$$\frac{d}{dx}\left[\int_a^x f\right] = \frac{d}{dx}\left[\int_a^x f^+ - \int_a^x f^-\right] = f^+ - f^- = f$$

Th 2. Let $f \in ABC[a,b]$. Then

$$\int_a^x f' = f(x) - f(a) \quad \forall\, x \in [a,b]$$

(and so $\int_a^b f' = f(b) - f(a)$)

On the other hand, let $f_n = f \wedge n$. Then $f_n \leq f$ and $f_n \uparrow f$ (3

$$F'(x) = \frac{d}{dx}\int_a^x f = \frac{d}{dx}\int_a^x (f - f_n + f_n) \overset{a.e}{=} \frac{d}{dx}\int_a^x (f - f_n) + \frac{d}{dx}\int_a^x f_n$$

$$\geq \frac{d}{dx}\int_a^x f_n \underset{\uparrow}{=} f_n(x)$$

by part I.

$\left(\text{as } f - f_n \geq 0 \text{ \& so } \int_a^x (f - f_n) \uparrow_x ;\ \text{thus } \frac{d}{dx}\int_a^x (f - f_n) \geq 0 \right)$

Consequently $F'(x) \geq f(x)$ a.e $x \in [a,b]$ & so

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,dx \quad \left(\text{as already noted, } \int_a^b F'(x) \leq \int_a^b f(x)\,dx \right)$$

By the "converse of monotonicity result", we arrive at

$F'(x) = f(x)$ a.e.

III. General case of $f \in \mathcal{L}_1[a,b]$. Then

$$f(x) = f^+(x) - f^-(x) \quad \forall x$$

$$\frac{d}{dx}\int_a^x f^+ = f^+ \text{ \& } \frac{d}{dx}\int_a^x f^- = f^- \quad \text{a.e on } [a,b]$$

(by part II).

Hence, for a.e x in $[a,b]$,

$$F'(x) = \frac{d}{dx}\int_a^x f = \frac{d}{dx}\int_a^x (f^+ - f^-) = \frac{d}{dx}\int_a^x f^+ - \frac{d}{dx}\int_a^x f^- = f^+ - f^- = f$$

Proof. Let

$$\Phi(x) = f(x) - \int_a^x f' \quad \forall\, x \in [a,b].$$

Then $\Phi \in ABC[a,b]$ and for a.e x in $[a,b]$

$$\Phi'(x) = f'(x) - \frac{d}{dx}\int_a^x f'$$

$$= f'(x) - f'(x)$$

$$= 0 \quad \left(\begin{array}{l} \text{since } f \in ABC[a,b], \\ f'(x) \text{ exists in } \mathbb{R} \text{ a.e } x \text{ in } [a,b] \\ \text{\& } f' \in \mathcal{L}_1[a,b] \text{ so Th 1 applicable} \end{array} \right)$$

By the following lemma, Φ is a constant

Lemma. Let $\Phi \in ABC[a,b]$ with $\Phi' = 0$ a.e on $[a,b]$. Then $\Phi \equiv$ constant 0 on $[a,b]$.

Proof. Need only show that $\Phi(a) = \Phi(b)$ (why?). To do this let $\varepsilon > 0$ and take $\delta > 0$ as from the definition of ABC corresponding to ε. For the set

$$E := \{x \in (a,b) : \Phi'(x) = 0\} \quad \left(\text{of mea} = m([a,b])\right),$$

we have a V-cover of the form $[x, x'] \subseteq (a,b)$ with $x \in E$ and $\left|\frac{f(x) - f(x')}{x - x'}\right| < \varepsilon$ and so $\exists$ finitely many disjoint $[x_i, x_i']$ $(i = 1, 2 \cdots N)$ of them s.t.

$$m\left(E \setminus \bigcup_{i=1}^{N} [x_i, x_i']\right) < \delta$$

(we may assume that $x_i' < x_{i+1}$ $\forall\, i = 1, 2, \cdots, N-1$)

Then $m\left([a,b] \setminus \bigcup_{i=1}^{N} [x_i, x_i']\right) < \delta$ and so

$$[a, x_1) \cup (x_1', x_2) \cup (x_2', x_3) \cup \cdots \cup (x_N', b]$$

are of total length $< \delta$ and so

$$\sum_{i=1}^{N+1} |f(x_{i-1}') - f(x_i)| < \varepsilon \qquad \left(\begin{array}{l} x_0' := a \\ x_{N+1} := b \end{array} \right)$$

while

$$\sum_{i=1}^{N} |f(x_i) - f(x_i')| < \sum_{i=1}^{N} \varepsilon |x_i - x_i'| \leq \varepsilon (b-a).$$

Combining these two inequality and making use of the triangle inequality it follows that

$$|f(a) - f(b)| < \varepsilon + \varepsilon(b-a)$$

and so $f(a) = f(b)$ as required ~~to~~ show.