MATH1010 University Mathematics

Department of Mathematics
The Chinese University of Hong Kong

1. **Limits**
   - Sequences
     - Limits of sequences
     - Squeeze theorem
     - Monotone convergence theorem
   - Limits of functions
     - Limits of functions
     - Exponential, logarithmic and trigonometric functions
   - Continuity of functions

2. **Differentiation**
   - Derivatives
     - Differentiable functions
     - Rules of differentiation
     - Second and higher derivatives
   - Mean value theorem
     - Mean value theorem
   - Application of Differentiation
     - L'Hopital's rule
     - Taylor series
     - Curve sketching

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Limits of sequences

Definition (Infinite sequence of real numbers)

An **infinite sequence of real numbers** is defined by a function from the set of positive integers $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ to the set of real numbers $\mathbb{R}$.

Example (Arithmetic sequence)

An **arithmetic sequence** is a sequence a_n such that $a_{n+1} - a_n = d$ is a constant independent of n. The constant d is called the **common difference**. The n-th term of the sequence is

$$a_n = a_1 + (n-1)d.$$

Sequence	a_1	d	a_n
$1, 3, 5, 7, 9, \dots$	1	2	$a_n = 2n - 1$
$-4, -1, 2, 5, 8, \dots$	-4	3	$a_n = 3n - 7$
$19, 12, 5, -2, -9, \dots$	19	-7	$a_n = 26 - 7n$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example (Geometric sequence)

A **geometric sequence** is a sequence a_n such that $a_{n+1} = ra_n$ for any n where r is a constant. The constant r is called the **common ratio**. The n-th term of the sequence is

$$a_n = a_1 r^{n-1}.$$

Sequence	a_1	r	a_n
$1, 2, 4, 8, 16, \ldots$	1	2	$a_n = 2^{n-1}$
$18, 6, 2, \dfrac{2}{3}, \dfrac{2}{9}, \ldots$	18	$\dfrac{1}{3}$	$a_n = \dfrac{54}{3^n}$
$12, -6, 3, -\dfrac{3}{2}, \dfrac{3}{4}, \ldots$	12	$-\dfrac{1}{2}$	$a_n = \dfrac{(-1)^{n-1}24}{2^n}$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let r and d be real numbers. Let a_n, $n = 0, 1, 2, \cdots$, be a sequence which satisfies

$$a_{n+1} = r a_n + d, \quad \text{for } n \geq 0.$$

Then

$$a_n = a_0 r^n + \left(\frac{r^n - 1}{r - 1} \right) d.$$

For $a_0 = 1000$, $r = 1.003$, $d = -10$, we have

n	0	1	2	3	4	5
a_n	1000	993	985.98	978.94	971.87	964.79
n	24	$\cdots$	60	$\cdots$	119	120
a_n	826.07	$\cdots$	540.58	$\cdots$	0.70	-9.30

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example (Fibonacci sequence)

The **Fibonacci sequence** is the sequence F_n which satisfies

$$\begin{cases} F_{n+2} = F_{n+1} + F_n, \text{ for } n \geq 1 \\ F_1 = F_2 = 1 \end{cases}$$

The first few terms of F_n are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

The value of F_n can be calculated by

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition (Limit of sequence)

1. Suppose there exists real number L such that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $|a_n - L| < \epsilon$. Then we say that a_n is **convergent**, or a_n **converges to** L, and write

$$\lim_{n \to \infty} a_n = L.$$

Otherwise we say that a_n is **divergent**.

2. Suppose for any $M > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $a_n > M$. Then we say that a_n **tends to** $+\infty$ as n tends to infinity, and write

$$\lim_{n \to \infty} a_n = +\infty.$$

We define a_n **tends to** $-\infty$ in a similar way. Note that a_n is divergent if it tends to $\pm\infty$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example (Convergent and divergent sequence)

Sequence	Convergent	Limit
$2.9, 2.99, 2.999, 2.9999, ...$	✓	3
$\dfrac{11}{21}, \dfrac{101}{201}, \dfrac{1001}{2001}, \dfrac{10001}{20001}, ...$	✓	$\dfrac{1}{2}$
$1, -\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{4}, \dfrac{1}{5}, ...$	✓	0
$2, 2, 2, 2, 2, ...$	✓	2
$1, 0, \dfrac{1}{2}, 0, \dfrac{1}{3}, 0, \dfrac{1}{4}, ...$	✓	0
$1, 0, 1, 0, 1, 0, ...$	$\times$	$-$
$1, 11, 111, 1111, 11111, ...$	$\times$	$+\infty$
$1, -3, 5, -7, 9, ...$	$\times$	$-$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example (Intuitive meaning of limits of infinite sequences)

a_n	First few terms	Limit
$\dfrac{1}{n^2}$	$1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dots$	0
$\dfrac{n}{n+1}$	$\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dots$	1
$(-1)^{n+1}$	$1, -1, 1, -1, \dots$	does not exist
$2n$	$2, 4, 6, 8, \dots$	does not exist$/ + \infty$
$\left(1 + \dfrac{1}{n}\right)^n$	$2, \dfrac{9}{4}, \dfrac{64}{27}, \dfrac{625}{256}, \dots$	$e \approx 2.71828$
$\dfrac{F_{n+1}}{F_n}$	$1, 2, \dfrac{3}{2}, \dfrac{5}{3}, \dots$	$\dfrac{1+\sqrt{5}}{2} \approx 1.61803$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition (Monotonic sequence)

1. We say that a_n is **monotonic increasing** (**decreasing**) if for any $m < n$, we have $a_m \leq a_n$ ($a_m \geq a_n$). We say that a_n is **monotonic** if a_n is either monotonic increasing or monotonic decreasing.

2. We say that a_n is **strictly increasing** (**decreasing**) if for any $m < n$, we have $a_m < a_n$ ($a_m > a_n$).

Definition (Bounded sequence)

We say that a_n is **bounded** if there exists real number M such that $|a_n| < M$ for any $n \in \mathbb{N}$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example (Monotonicity and boundedness)

Sequence	Monotonic	Strictly monotonic	Bounded
$3, 3, 3, 3, 3, ...$	✓	✗	✓
$1, 1, 2, 2, 3, 3, 4, 4, ...$	✓	✗	✗
$7, -2, 7, -2, 7, -2, ...$	✗	✗	✓
$2.7, 2.77, 2.777, 2.7777, ...$	✓	✓	✓
$1, 0, 2, 0, 3, 0, 4, 0, ...$	✗	✗	✗
$-1, -2, -3, -4, ...$	✓	✓	✗
$0.001, 0.002, 0.003, 0.004, ...$	✓	✓	✗
$1000, \dfrac{1000}{2}, \dfrac{1000}{3}, \dfrac{1000}{4}, ...$	✓	✓	✓

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example (Bounded and monotonic sequence)

a_n	Terms	Bounded	Monotonic	Convergent (Limit)
$\dfrac{1}{n^2}$	$1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dots$	$\checkmark$	$\checkmark$	$\checkmark$ (0)
$1 - \dfrac{(-1)^n}{n}$	$2, \dfrac{1}{2}, \dfrac{4}{3}, \dfrac{3}{4}, \dots$	$\checkmark$	$\times$	$\checkmark$ (1)
n^2	$1, 4, 9, 16, \dots$	$\times$	$\checkmark$	$\times$
$1 - (-1)^n$	$2, 0, 2, 0, \dots$	$\checkmark$	$\times$	$\times$
$(-1)^n n$	$-1, 2, -3, 4, \dots$	$\times$	$\times$	$\times$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

*If a_n is **convergent**, then a_n is **bounded**.*

$$\textbf{Convergent} \Rightarrow \textbf{Bounded}$$

Note that the converse of the above statement is not correct.

$$\textbf{Bounded} \not\Rightarrow \textbf{Convergent}$$

The following theorem is very important and we will discuss it in details later.

Theorem (Monotone convergence theorem)

*If a_n is **bounded** and **monotonic**, then a_n is **convergent**.*

$$\textbf{Bounded } and \textbf{ Monotonic} \Rightarrow \textbf{Convergent}$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*

$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b.$$

Answer: **T**

Limits | **Sequences**
Differentiation | Limits of functions
Integration | Continuity of functions

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$ and c is a real number. Then

$$\lim_{n\to\infty} ca_n = ca.$$

Answer: **T**

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim_{n \to \infty} a_n = a$ *and* $\lim_{n \to \infty} b_n = b$, *then*

$$\lim_{n \to \infty} a_n b_n = ab.$$

Answer: **T**

Limits   **Sequences**
Differentiation   Limits of functions
Integration   Continuity of functions

Exercise (True or False)

If $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$, then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Answer: **F**

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b \neq 0$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Answer: **T**

Limits
Differentiation
Integration
Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim\limits_{n \to \infty} a_n = 0$, then

$$\lim_{n \to \infty} a_n b_n = 0.$$

Answer: **F**

Example

For $a_n = \dfrac{1}{n}$ and $b_n = n$, we have $\lim\limits_{n \to \infty} a_n = 0$ but

$$\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} \frac{1}{n} \cdot n = \lim_{n \to \infty} 1 = 1 \neq 0.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim\limits_{n\to\infty} a_n = 0$ *and* b_n *is* **convergent**, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **T**

Proof.

$$
\begin{aligned}
\lim_{n\to\infty} a_n b_n &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n \\
&= 0
\end{aligned}
$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim\limits_{n \to \infty} a_n = 0$ and b_n is **bounded**, then

$$\lim_{n \to \infty} a_n b_n = 0.$$

Answer: **T**

Caution! The previous proof does not work.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If a_n and b_n are divergent, then $a_n + b_n$ is divergent.

Answer: **F**

Example

The sequences $a_n = n$ and $b_n = -n$ are divergent but $a_n + b_n = 0$ converges to 0.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim_{n\to\infty} b_n = +\infty$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **F**

Example

For $a_n = n^2$ and $b_n = n$, we have $\lim_{n\to\infty} b_n = +\infty$ but

$\dfrac{a_n}{b_n} = \dfrac{n^2}{n} = n$ is divergent.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

*If a_n is **bounded** and $\lim\limits_{n\to\infty} b_n = \pm\infty$, then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **T**

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

Suppose a_n and b_n are convergent sequences such that $a_n < b_n$ for any n. Then

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n.$$

Answer: **F**

Example

The sequences $a_n = 0$ and $b_n = \dfrac{1}{n}$ satisfy $a_n < b_n$ for any n.
However

$$\lim_{n\to\infty} a_n \not< \lim_{n\to\infty} b_n$$

because both of them are 0.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

Suppose a_n and b_n are convergent sequences such that $a_n \leq b_n$ for any n. Then

$$\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n.$$

Answer: **T**

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If a_n is convergent, then

$$\lim_{n \to \infty} (a_{n+1} - a_n) = 0.$$

Answer: **T**

Limits | **Sequences**
Differentiation | Limits of functions
Integration | Continuity of functions

Exercise (True or False)

If $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = 0$, then a_n is convergent.

Answer: **F**

Example

Let $a_n = \sqrt{n}$. Then $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = 0$ and a_n is divergent.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exercise (True or False)

If $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=0$ and a_n is bounded, then a_n is convergent.

Answer: **F**

Example

$$0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{6}, \frac{2}{6}, \cdots$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let a_n, b_n be two sequences such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and c be a real number. Then

1. $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$

2. $\lim_{n\to\infty} ca_n = ca$

3. $\lim_{n\to\infty} a_n b_n = ab$

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$ *if $b \neq 0$.*

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let a_n be a sequence such that $\lim\limits_{n\to\infty} a_n = a$. Then

1. *for any positive integer k, $\lim\limits_{n\to\infty} a_{n+k} = a$.*

2. $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = 0$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let a be a real number.

$$\lim_{n \to \infty} a^n = \begin{cases} 0, & \text{if } -1 < a < 1 \\ 1, & \text{if } a = 1 \\ \text{does not exist}, & \text{if } a \leq -1 \text{ or } a > 1 \end{cases}.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let $a \neq 0$ and $r \neq 1$ be real numbers. Let

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Then

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

and

$$
\begin{aligned}
\lim_{n \to \infty} s_n &= \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} \\
&= \begin{cases} \dfrac{a}{1 - r}, & \text{if } -1 < r < 1 \\ \text{does not exist}, & \text{otherwise} \end{cases}.
\end{aligned}
$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$$\lim_{n \to \infty} \frac{2n-5}{3n+1} = \lim_{n \to \infty} \frac{2 - \frac{5}{n}}{3 + \frac{1}{n}}$$
$$= \frac{2-0}{3+0}$$
$$= \frac{2}{3}$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$$\lim_{n \to \infty} \frac{n^3 - 2n + 7}{4n^3 + 5n^2 - 3} = \lim_{n \to \infty} \frac{1 - \frac{2}{n^2} + \frac{7}{n^3}}{4 + \frac{5}{n} - \frac{3}{n^3}}$$

$$= \frac{1}{4}$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$$\lim_{n \to \infty} \frac{3n - \sqrt{4n^2 + 1}}{3n + \sqrt{9n^2 + 1}} = \lim_{n \to \infty} \frac{3 - \frac{\sqrt{4n^2+1}}{n}}{3 + \frac{\sqrt{9n^2+1}}{n}}$$

$$= \lim_{n \to \infty} \frac{3 - \sqrt{4 + \frac{1}{n^2}}}{3 + \sqrt{9 + \frac{1}{n^2}}}$$

$$= \frac{1}{6}$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$$\lim_{n \to \infty} (n - \sqrt{n^2 - 4n + 1})$$

$$= \lim_{n \to \infty} \frac{(n - \sqrt{n^2 - 4n + 1})(n + \sqrt{n^2 - 4n + 1})}{n + \sqrt{n^2 - 4n + 1}}$$

$$= \lim_{n \to \infty} \frac{n^2 - (n^2 - 4n + 1)}{n + \sqrt{n^2 - 4n + 1}}$$

$$= \lim_{n \to \infty} \frac{4n - 1}{n + \sqrt{n^2 - 4n + 1}}$$

$$= \lim_{n \to \infty} \frac{4 - \frac{1}{n}}{1 + \sqrt{1 - \frac{4}{n} + \frac{1}{n^2}}}$$

$$= 2$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\ln(n^4+1)}{\ln(n^3+1)} &= \lim_{n\to\infty} \frac{\ln(n^4(1+\frac{1}{n^4}))}{\ln(n^3(1+\frac{1}{n^3}))} \\
&= \lim_{n\to\infty} \frac{\ln n^4 + \ln(1+\frac{1}{n^4})}{\ln n^3 + \ln(1+\frac{1}{n^3})} \\
&= \lim_{n\to\infty} \frac{4\ln n + \ln(1+\frac{1}{n^4})}{3\ln n + \ln(1+\frac{1}{n^3})} \\
&= \lim_{n\to\infty} \frac{4 + \frac{\ln(1+\frac{1}{n^4})}{\ln n}}{3 + \frac{\ln(1+\frac{1}{n^3})}{\ln n}} \\
&= \frac{4}{3}
\end{aligned}
$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Squeeze theorem

Theorem (Squeeze theorem)

Suppose a_n, b_n, c_n are sequences such that $a_n \leq b_n \leq c_n$ for any n and $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} c_n = L$. Then b_n is convergent and

$$\lim_{n \to \infty} b_n = L.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

If a_n is bounded and $\lim_{n \to \infty} b_n = 0$, then $\lim_{n \to \infty} a_n b_n = 0$.

Proof.

Since a_n is bounded, there exists M such that $-M < a_n < M$ for any n. Thus

$$-M|b_n| < a_n b_n < M|b_n|$$

for any n. Now

$$\lim_{n \to \infty} (-M|b_n|) = \lim_{n \to \infty} M|b_n| = 0.$$

Therefore by squeeze theorem, we have

$$\lim_{n \to \infty} a_n b_n = 0.$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Find $\lim\limits_{n\to\infty} \dfrac{\sqrt{n} + (-1)^n}{\sqrt{n} - (-1)^n}$.

Solution

Since $(-1)^n$ is bounded and $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$, we have

$\lim\limits_{n\to\infty} \dfrac{(-1)^n}{\sqrt{n}} = 0$ and therefore

$$\lim_{n\to\infty} \frac{\sqrt{n} + (-1)^n}{\sqrt{n} - (-1)^n} = \lim_{n\to\infty} \frac{1 + \frac{(-1)^n}{\sqrt{n}}}{1 - \frac{(-1)^n}{\sqrt{n}}}$$
$$= 1$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Show that $\lim_{n\to\infty} \dfrac{4^n}{n!} = 0$.

Proof.

Observe that for any $n \geq 4$,

$$0 < \frac{4^n}{n!} = \frac{4^3}{3!} \left(\frac{4}{4} \cdot \frac{4}{5} \cdot \frac{4}{6} \cdots \frac{4}{n-1} \right) \frac{4}{n} \leq \frac{4^3}{3!} \cdot \frac{4}{n} = \frac{128}{3n}$$

and $\lim_{n\to\infty} \dfrac{128}{3n} = 0$. By squeeze theorem, we have

$$\lim_{n\to\infty} \frac{4^n}{n!} = 0.$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let $a_n = \dfrac{1}{n^3 + 1^2} + \dfrac{1}{n^3 + 2^2} + \dfrac{1}{n^3 + 3^2} + \cdots + \dfrac{1}{n^3 + n^2}$. Find $\lim\limits_{n \to \infty} a_n$.

Solution

Observe that for any n,

$$\frac{n}{n^3 + n^2} \le \frac{1}{n^3 + 1^2} + \frac{1}{n^3 + 2^2} + \frac{1}{n^3 + 3^2} + \cdots + \frac{1}{n^3 + n^2} \le \frac{n}{n^3 + 1}$$

and

$$
\begin{aligned}
\lim_{n \to \infty} \frac{n}{n^3 + n^2} &= \lim_{n \to \infty} \frac{1}{n^2 + n} = 0 \\
\lim_{n \to \infty} \frac{n}{n^3 + 1} &= \lim_{n \to \infty} \frac{1}{n^2 + \frac{1}{n}} = 0.
\end{aligned}
$$

By squeeze theorem, we have

$$\lim_{n \to \infty} \left(\frac{1}{n^3 + 1^2} + \frac{1}{n^3 + 2^2} + \frac{1}{n^3 + 3^2} + \cdots + \frac{1}{n^3 + n^2} \right) = 0.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Monotone convergence theorem

Theorem (Monotone convergence theorem)

*If a_n is **bounded** and **monotonic**, then a_n is **convergent**.*

Bounded *and* **Monotonic** $\Rightarrow$ **Convergent**

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let a_n be the sequence defined by the recursive relation
$$\begin{cases} a_{n+1} = \sqrt{a_n + 1} \text{ for } n \geq 1 \\ a_1 = 1 \end{cases}.$$
Find $\lim\limits_{n \to \infty} a_n$.

n	a_n
1	1
2	1.414213562
3	1.553773974
4	1.598053182
5	1.611847754
10	1.618016542
15	1.618033940

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Solution

Suppose $\lim\limits_{n \to \infty} a_n = a$. Then

$$
\begin{aligned}
\lim_{n \to \infty} a_{n+1} &= \lim_{n \to \infty} \sqrt{a_n + 1} \\
a &= \sqrt{a + 1} \\
a^2 &= a + 1 \\
a^2 - a - 1 &= 0 \\
a &= \frac{1 + \sqrt{5}}{2} \text{ or } \frac{1 - \sqrt{5}}{2}.
\end{aligned}
$$

It is obvious that $a > 0$. Therefore

$$
a = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887
$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Solution

The above solution is not complete. The solution is valid only after we have proved that $\lim_{n \to \infty} a_n$ exists and is positive. This can be done by using **monotone convergence theorem**. *We are going to show that a_n is* **bounded** *and* **monotonic**.

Boundedness

We prove that $1 \le a_n < 2$ for all $n \ge 1$ by induction.

(Base case) When $n = 1$, we have $a_1 = 1$ and $1 \le a_1 < 2$.

(Induction step) Assume that $1 \le a_k < 2$. Then

$$
\begin{aligned}
a_{k+1} &= \sqrt{a_k + 1} \ge \sqrt{1 + 1} > 1 \\
a_{k+1} &= \sqrt{a_k + 1} < \sqrt{2 + 1} < 2
\end{aligned}
$$

Thus $1 \le a_n < 2$ for any $n \ge 1$ which implies that a_n is bounded.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Solution

Monotonicity: We prove that $a_{n+1} > a_n$ for any $n \geq 1$ by induction.
(Base case) When $n = 1$, $a_1 = 1$, $a_2 = \sqrt{2}$ and thus $a_2 > a_1$.
(Induction step) Assume that

$$a_{k+1} > a_k \text{ (Induction hypothesis)}.$$

Then

$$
\begin{aligned}
a_{k+2} &= \sqrt{a_{k+1} + 1} > \sqrt{a_k + 1} \text{ (by induction hypothesis)} \\
&= a_{k+1}
\end{aligned}
$$

This completes the induction step and thus a_n is strictly increasing.
We have proved that a_n is bounded and strictly increasing. Therefore a_n
is convergent by monotone convergence theorem. Since $a_n \geq 1$ for any
n, we have $\lim\limits_{n\to\infty} a_n \geq 1$ is positive.

This completes that proof that $\lim\limits_{n\to\infty} a_n = \dfrac{1 + \sqrt{5}}{2}$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let a_n be a sequence defined by

$$\begin{cases} a_{n+1} = 2a_n - a_n^2, & \text{for } n \geq 1 \\ a_1 = \dfrac{1}{2} \end{cases}$$

1. Prove that $a_n \leq 1$ for any positive integer n.

2. Prove that a_n is monotonic increasing.

3. Find $\lim\limits_{n \to \infty} a_n$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Solution

1. *Observe that $a_1 = \frac{1}{2} < 1$ and for any $n \geq 2$, we have*

$$a_n = 2a_{n-1} - a_{n-1}^2 = -(a_{n-1} - 1)^2 + 1 \leq 1.$$

   Therefore $a_n \leq 1$ for any positive integer n.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Solution

2. *We prove that $a_{n+1} - a_n \geq 0$ for any n by induction on n.*
   (Base case) When $n = 1$, $a_2 - a_1 = \frac{3}{4} - \frac{1}{2} > 0$. (Induction step)
   Assume that $a_{k+1} - a_k \geq 0$. Then

$$
\begin{aligned}
a_{k+2} - a_{k+1} &= (2a_{k+1} - a_{k+1}^2) - (2a_k - a_k^2) \\
&= 2(a_{k+1} - a_k) - (a_{k+1}^2 - a_k^2) \\
&= 2(a_{k+1} - a_k) - (a_{k+1} + a_k)(a_{k+1} - a_k) \\
&= (2 - (a_{k+1} + a_k))(a_{k+1} - a_k)
\end{aligned}
$$

   Since $a_k, a_{k+1} \leq 1$ by (1) and $a_{k+1} - a_k \geq 0$ by induction hypothesis, we have $a_{k+2} - a_{k+1} \geq 0$. This completes the induction step and we conclude that a_n is monotonic increasing.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Solution

3. *Since $a_n \leq 1$ is bounded and a_n is monotonic increasing, a_n is convergent by monotone convergence theorem. Let $\lim_{n \to \infty} a_n = a$. Then*

$$
\begin{aligned}
\lim_{n \to \infty} a_{n+1} &= \lim_{n \to \infty} (2a_n - a_n^2) \\
a &= 2a - a^2 \\
a^2 - a &= 0 \\
a(a-1) &= 0 \\
a &= 1 \text{ or } 0
\end{aligned}
$$

Since $a_n \geq a_1 = \frac{1}{2}$ for any n, we have $a \geq \frac{1}{2} > 0$. Therefore $a = 1$ and we proved that $\lim_{n \to \infty} a_n = 1$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Let $a_n = \dfrac{F_{n+1}}{F_n}$ where F_n is the Fibonacci's sequence defined by
$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_1 = F_2 = 1 \end{cases}.$$
Find $\lim\limits_{n \to \infty} a_n$.

n	a_n
1	1
2	2
3	1.5
4	1.666666666
5	1.6
10	1.618181818
15	1.618032787
20	1.618033999

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

For any $n \geq 1$,

1. $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$
2. $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$

Proof

1. *When $n = 1$, we have $F_3 F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2$. Assume*

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}.$$

Then

$$
\begin{aligned}
F_{k+3}F_{k+1} - F_{k+2}^2 &= (F_{k+2} + F_{k+1})F_{k+1} - F_{k+2}^2 \\
&= F_{k+2}(F_{k+1} - F_{k+2}) + F_{k+1}^2 \\
&= -F_{k+2}F_k + F_{k+1}^2 \\
&= (-1)^{k+2} \text{ (by induction hypothesis)}
\end{aligned}
$$

Therefore $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$ for any $n \geq 1$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Proof.

The proof for the second statement is basically the same. When $n = 1$, we have $F_4 F_1 - F_3 F_2 = 3 \cdot 1 - 2 \cdot 1 = 1 = (-1)^2$. Assume

$$F_{k+3} F_k - F_{k+2} F_{k+1} = (-1)^{k+1}.$$

Then

$$
\begin{aligned}
F_{k+4} F_{k+1} - F_{k+3} F_{k+2} &= (F_{k+3} + F_{k+2}) F_{k+1} - F_{k+3} F_{k+2} \\
&= F_{k+3}(F_{k+1} - F_{k+2}) + F_{k+2} F_{k+1} \\
&= -F_{k+3} F_k + F_{k+2} F_{k+1} \\
&= -(-1)^{k+1} \text{ (by induction hypothesis)} \\
&= (-1)^{k+2}
\end{aligned}
$$

Therefore $F_{n+3} F_n - F_{n+2} F_{n+1} = (-1)^{n+1}$ for any $n \geq 1$. $\qquad \square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let $a_n = \dfrac{F_{n+1}}{F_n}$.

1. The sequence $a_1, a_3, a_5, a_7, \cdots$, is strictly increasing.

2. The sequence $a_2, a_4, a_6, a_8, \cdots$, is strictly decreasing.

Proof.

For any $k \geq 1$, we have

$$
\begin{aligned}
a_{2k+1} - a_{2k-1} &= \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{2k}}{F_{2k-1}} = \frac{F_{2k+2}F_{2k-1} - F_{2k+1}F_{2k}}{F_{2k+1}F_{2k-1}} \\
&= \frac{(-1)^{2k}}{F_{2k+1}F_{2k-1}} = \frac{1}{F_{2k+1}F_{2k-1}} > 0
\end{aligned}
$$

Therefore $a_1, a_3, a_5, a_7, \cdots$, is strictly increasing. The second statement can be proved in a similar way. $\qquad \square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

$$\lim_{k \to \infty} (a_{2k+1} - a_{2k}) = 0$$

Proof.

For any $k \geq 1$,

$$
\begin{aligned}
a_{2k+1} - a_{2k} &= \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{2k+1}}{F_{2k}} \\
&= \frac{F_{2k+2}F_{2k} - F_{2k+1}^2}{F_{2k+1}F_{2k}} = \frac{1}{F_{2k+1}F_{2k}}
\end{aligned}
$$

Therefore

$$\lim_{k \to \infty} (a_{2k+1} - a_{2k}) = \lim_{k \to \infty} \frac{1}{F_{2k+1}F_{2k}} = 0.$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

Proof

First we prove that $a_n = \frac{F_{n+1}}{F_n}$ is convergent.
a_n is bounded. ($1 \le a_n \le 2$ for any n.)
a_{2k+1} and a_{2k} are convergent. (They are bounded and monotonic.)

$$\lim_{k\to\infty} (a_{2k+1} - a_{2k}) = 0 \Rightarrow \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{2k}$$

It follows that a_n is convergent and

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{2k}.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Proof.

To evaluate the limit, suppose $\lim\limits_{n \to \infty} \dfrac{F_{n+1}}{F_n} = L$. Then

$$L = \lim_{n \to \infty} \frac{F_{n+2}}{F_{n+1}} = \lim_{n \to \infty} \frac{F_{n+1} + F_n}{F_{n+1}} = \lim_{n \to \infty} \left(1 + \frac{F_n}{F_{n+1}} \right) = 1 + \frac{1}{L}$$

$$L^2 - L - 1 = 0$$

By solving the quadratic equation, we have

$$L = \frac{1 + \sqrt{5}}{2} \text{ or } \frac{1 - \sqrt{5}}{2}.$$

We must have $L \geq 1$ since $a_n \geq 1$ for any n. Therefore

$$L = \frac{1 + \sqrt{5}}{2}.$$

□

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Remarks

The limit can be calculated directly using the formula

$$
\begin{aligned}
F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \\
&= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)
\end{aligned}
$$

where

$$
\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}
$$

are the roots of the quadratic equation

$$
x^2 - x - 1 = 0.
$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let

$$
\begin{aligned}
a_n &= \left(1 + \frac{1}{n}\right)^n \\
b_n &= \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}
\end{aligned}
$$

Then

1. $a_n < b_n$ *for any* $n > 1$.

2. a_n *and* b_n *are convergent and*

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n
$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$b_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

n	a_n	b_n
1	2	2
5	2.48832	2.716666666666
10	2.593742	2.718281801146
100	2.704813	2.718281828459
100000	2.718268	2.718281828459

The limit of the two sequences is the important Euler's number

$$e \approx 2.71828\,18284\,59045\,23536\ldots.$$

which is also known as the Napier's constant.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition (Convergence of infinite series)

We say that an **infinite series**

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is **convergent** if the sequence of partial sums
$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$ is convergent. If the infinite series
is convergent, then we define

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Limits of functions

Definition (Function)

A real valued **function** on a subset $D \subset \mathbb{R}$ is a real value $f(x)$ assigned to each of the values $x \in D$. The set D is called the **domain** of the function.

Given an expression $f(x)$ in x, the domain D is understood to be taken as the set of all real numbers x such that $f(x)$ is defined. This is called the maximum domain of definition of $f(x)$.

Definition (Graph of function)

Let $f(x)$ is a real valued function. The graph of $f(x)$ is the set

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition

Let $f(x)$ be a real valued function and D be its domain. We say that $f(x)$ is

1. **injective** if for any $x_1, x_2 \in D$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.
2. **surjective** if for any real number $y \in \mathbb{R}$, there exists $x \in D$ such that $f(x) = y$.
3. **bijective** if $f(x)$ is both injective and surjective.

Definition

Let $f(x)$ be a real valued function. We say that $f(x)$ is

1. **even** if $f(-x) = f(x)$ for any x.
2. **odd** if $f(-x) = -f(x)$ for any x.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$f(x)$	Domain	Injective	Surjective	Bijective	Even	Odd
$2x - 3$	$\mathbb{R}$	✓	✓	✓	×	×
$x^3 - 2x^2$	$\mathbb{R}$	×	✓	×	×	×
$\dfrac{1}{x}$	$x \neq 0$	✓	×	×	×	✓
$\dfrac{4x}{x^2 + 1}$	$\mathbb{R}$	×	×	×	×	✓
$\dfrac{x}{x^2 - 1}$	$x \neq \pm 1$	×	✓	×	×	✓
$x^2 - \dfrac{1}{x^2}$	$x \neq 0$	×	✓	×	✓	×
$\sqrt{4 - x^2}$	$-2 \leq x \leq 2$	×	×	×	✓	×
$\dfrac{1}{\sqrt{x + 4}}$	$x > -4$	✓	×	×	×	×

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

$$y = 2x - 3$$

$$y = x^3 - 2x^2$$

$$y = \frac{1}{x}$$

$$y = \frac{4x}{x^2 + 1}$$

Limits   Sequences
Differentiation   **Limits of functions**
Integration   Continuity of functions

$$y = \frac{x}{x^2 - 1}$$

$$y = x^2 - \frac{1}{x^2}$$

$$y = \sqrt{4 - x^2}$$

$$y = \frac{1}{\sqrt{x + 4}}$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
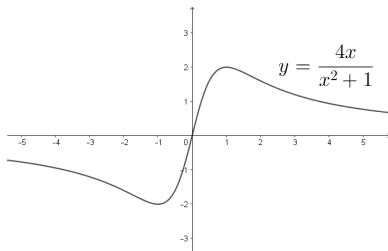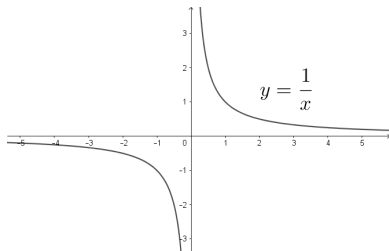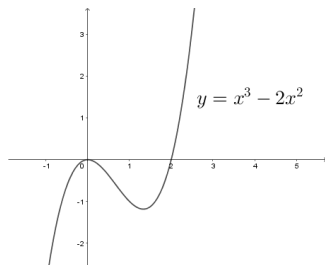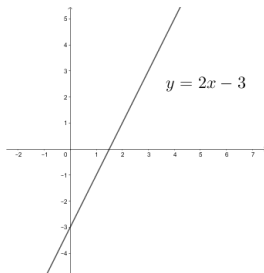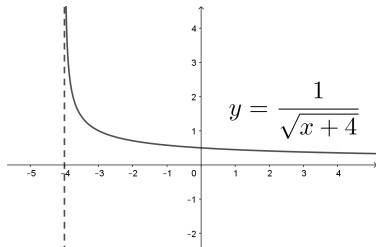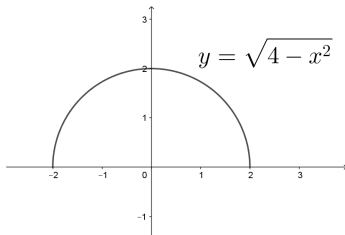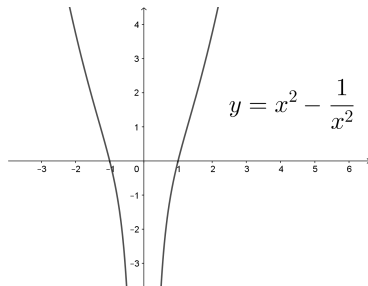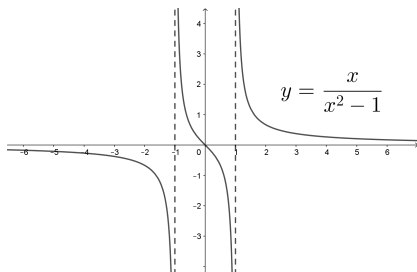
Definition (Limit of function)

Let $f(x)$ be a real valued function.

1 We say that a real number l is a limit of $f(x)$ at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - l| < \epsilon$$

and write

$$\lim_{x \to a} f(x) = l.$$

2 We say that a real number l is a limit of $f(x)$ at $+\infty$ if for any $\epsilon > 0$, there exists $R > 0$ such that

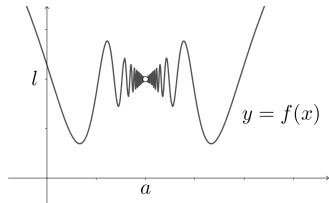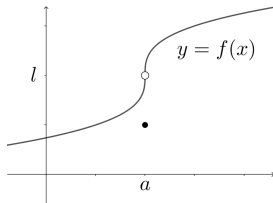$$\text{if } x > R, \text{ then } |f(x) - l| < \epsilon$$

and write

$$\lim_{x \to +\infty} f(x) = l.$$

The limit of $f(x)$ at $-\infty$ is defined similarly.

Limits
Differentiation
Integration

Sequences
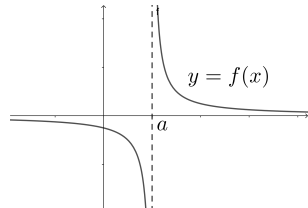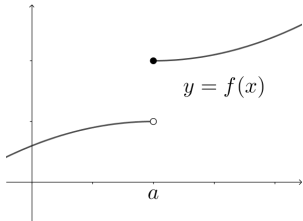**Limits of functions**
Continuity of functions

1. Note that for the limit of $f(x)$ at $x = a$ to exist, $f(x)$ may not be defined at $x = a$ and even if $f(a)$ is defined, the value of $f(a)$ does not affect the value of $\lim_{x \to a} f(x)$.

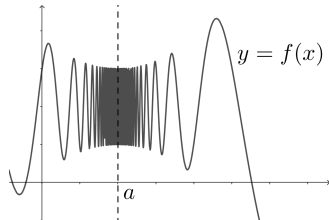2. The limit of $f(x)$ at $x = a$ may not exist. However the limit is unique if it exists.

Limits   Sequences
Differentiation   **Limits of functions**
Integration   Continuity of functions

$$\lim_{x \to a} f(x) = l$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

$$\lim_{x \to a} f(x)$$
does not exist

Limits
Differentiation
Integration
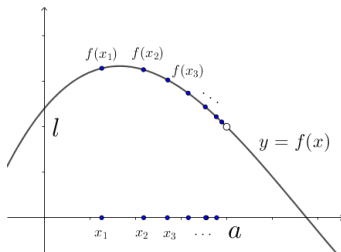
Sequences
Limits of functions
Continuity of functions
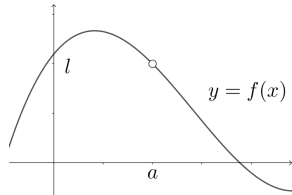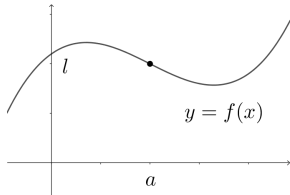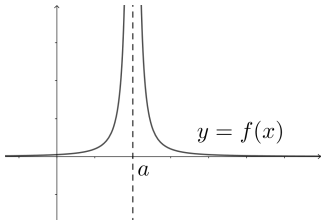
Theorem (Sequential criterion for limits of functions)

Let $f(x)$ be a real valued function. Then

$$\lim_{x \to a} f(x) = l$$

if and only if for any sequence x_n of real numbers with $\lim_{n \to \infty} x_n = a$, we have

$$\lim_{n \to \infty} f(x_n) = l.$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let $f(x)$, $g(x)$ be functions such that $\lim\limits_{x \to a} f(x)$, $\lim\limits_{x \to a} g(x)$ exist and c be a real number. Then

1. $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

2. $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$

3. $\lim\limits_{x \to a} f(x)g(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)$

4. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ *if* $\lim\limits_{x \to a} g(x) \neq 0$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem (Squeeze theorem)

Let $f(x), g(x), h(x)$ be real valued functions. Suppose

1. $f(x) \leq g(x) \leq h(x)$ *for any $x \neq a$ on a neighborhood of a, and*

2. $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$.

Then the limit of $g(x)$ at $x = a$ exists and $\lim_{x \to a} g(x) = l$.

Theorem

Suppose

1. $f(x)$ *is bounded, and*

2. $\lim_{x \to a} g(x) = 0$

Then $\lim_{x \to a} f(x)g(x) = 0$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Exponential, logarithmic and trigonometric functions

Definition (Exponential function)

The **exponential function** is defined for real number $x \in \mathbb{R}$ by

$$
\begin{aligned}
e^x &= \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\end{aligned}
$$

1. It can be proved that the two limits in the definition exist and converge to the same value for any real number x.

2. e^x is just a notation for the exponential function. One should not interpret it as 'e to the power x'.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

For any $x, y \in \mathbb{R}$, we have

$$e^{x+y} = e^x e^y.$$

Caution! One cannot use law of indices to prove the above identity. It is because e^x is just a notation for the exponential function and it does not mean 'e to the power x'. In fact we have not defined what a^x means when x is a real number which is not rational.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

1. $e^x > 0$ for any real number x.

2. e^x is strictly increasing.

Proof.

1. For any $x > 0$, we have $e^x > 1 + x > 1$. If $x < 0$, then

$$
\begin{aligned}
e^x e^{-x} &= e^{x+(-x)} = e^0 = 1 \\
e^x &= \frac{1}{e^{-x}} > 0
\end{aligned}
$$

since $e^{-x} > 1$. Therefore $e^x > 0$ for any $x \in \mathbb{R}$.

2. Let x, y be real numbers with $x < y$. Then $y - x > 0$ which implies $e^{y-x} > 1$. Therefore

$$
e^y = e^{x+(y-x)} = e^x e^{y-x} > e^x.
$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

$$y = e^x$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
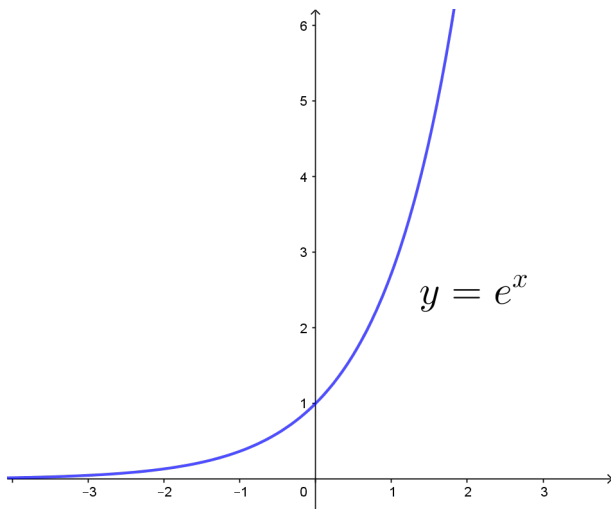
Definition (Logarithmic function)

The **logarithmic function** is the function $\ln : \mathbb{R}^+ \to \mathbb{R}$ defined for $x > 0$ by

$$y = \ln x \text{ if } e^y = x.$$

In other words, $\ln x$ is the inverse function of e^x.

It can be proved that for any $x > 0$, there exists unique real number y such that $e^y = x$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

1. $\ln xy = \ln x + \ln y$

2. $\ln \dfrac{x}{y} = \ln x - \ln y$

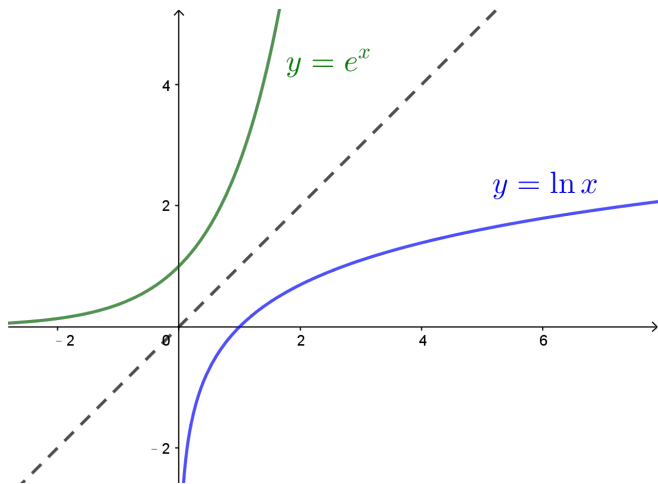3. $\ln x^n = n \ln x$ for any integer $n \in \mathbb{Z}$.

Proof.

1. Let $u = \ln x$ and $v = \ln y$. Then $x = e^u$, $y = e^v$ and we have

$$xy = e^u e^v = e^{u+v} = e^{\ln x + \ln y}$$

which means $\ln xy = \ln x + \ln y$.

Other parts can be proved similarly. □

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition (Cosine and sine functions)

The **cosine** and **sine** functions are defined for real number $x \in \mathbb{R}$ by the infinite series

$$
\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}
$$

1. When the sine and cosine are interpreted as trigonometric ratios, the angles are measured in radian. $(180^0 = \pi)$

2. The series for cosine and sine are convergent for any real number $x \in \mathbb{R}$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
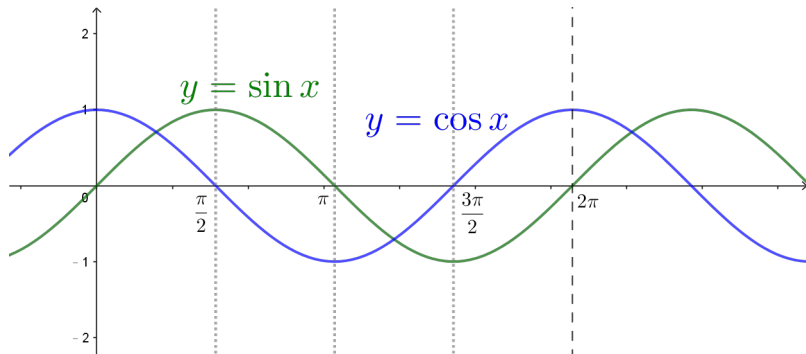
There are four more trigonometric functions namely tangent, cotangent, secant and cosecant functions. All of them are defined in terms of sine and cosine.

Definition (Trigonometric functions)

$$
\tan x = \frac{\sin x}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, \ k \in \mathbb{Z}
$$

$$
\cot x = \frac{\cos x}{\sin x}, \text{ for } x \neq k\pi, \ k \in \mathbb{Z}
$$

$$
\sec x = \frac{1}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, \ k \in \mathbb{Z}
$$

$$
\csc x = \frac{1}{\sin x}, \text{ for } x \neq k\pi, \ k \in \mathbb{Z}
$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem (Trigonometric identities)

1. $\cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$

2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y;$
   $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y;$
   $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x;$
   $\sin 2x = 2 \sin x \cos x;$
   $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

4. $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$
   $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$
   $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

5. $\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$
   $\cos x - \cos y = -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$
   $\sin x + \sin y = 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$
   $\sin x - \sin y = 2 \cos \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition (Hyperbolic function)

The **hyperbolic functions** are defined for $x \in \mathbb{R}$ by

$$
\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\
\sinh x &= \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots
\end{aligned}
$$

Limits

Differentiation

Integration

Sequences

Limits of functions

Continuity of functions

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
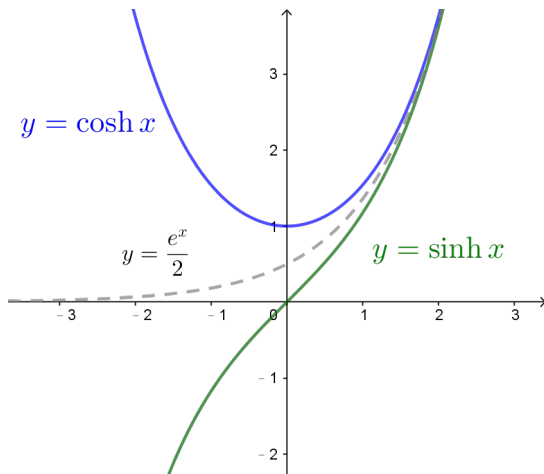
Theorem (Hyperbolic identities)

1. $\cosh^2 x - \sinh^2 x = 1$

2. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
   $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

3. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$;
   $\sinh 2x = 2\sinh x \cosh x$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

1. $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

2. $\displaystyle \lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$

3. $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Proof. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

For any $-1 < x < 1$ with $x \neq 0$, we have

$$
\begin{aligned}
\frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots \\
&\leq 1 + \frac{x}{2} + \left(\frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \cdots\right) = 1 + \frac{x}{2} + \frac{x^2}{2} \\
\frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots \\
&\geq 1 + \frac{x}{2} - \left(\frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \cdots\right) = 1 + \frac{x}{2} - \frac{x^2}{2}
\end{aligned}
$$

and $\displaystyle\lim_{x\to 0}(1 + \frac{x}{2} + \frac{x^2}{2}) = \lim_{x\to 0}(1 + \frac{x}{2} - \frac{x^2}{2}) = 1$. Therefore $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. $\square$

Limits   Sequences
Differentiation   **Limits of functions**
Integration   Continuity of functions

Figure: $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
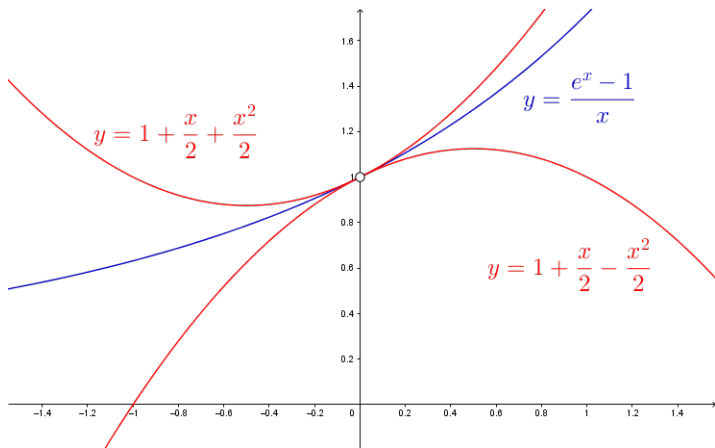
Proof. $\lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x} = 1$.

Let $y = \ln(1+x)$. Then

$$
\begin{aligned}
e^y &= 1 + x \\
x &= e^y - 1
\end{aligned}
$$

and $x \to 0$ as $y \to 0$. We have

$$
\begin{aligned}
\lim_{x \to 0} \frac{\ln(1+x)}{x} &= \lim_{y \to 0} \frac{y}{e^y - 1} \\
&= 1
\end{aligned}
$$

$\square$

Note that the first part implies $\lim\limits_{y \to 0}(e^y - 1) = 0$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Proof. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

Note that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots.$$

For any $-1 < x < 1$ with $x \neq 0$, we have

$$\frac{\sin x}{x} = 1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} \right) - \left(\frac{x^6}{7!} - \frac{x^8}{9!} \right) - \cdots \leq 1$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \left(\frac{x^4}{5!} - \frac{x^6}{7!} \right) + \left(\frac{x^8}{9!} - \frac{x^{10}}{11!} \right) + \cdots \geq 1 - \frac{x^2}{6}$$

and $\lim_{x \to 0} 1 = \lim_{x \to 0} (1 - \dfrac{x^2}{6}) = 1$. Therefore

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Figure: $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
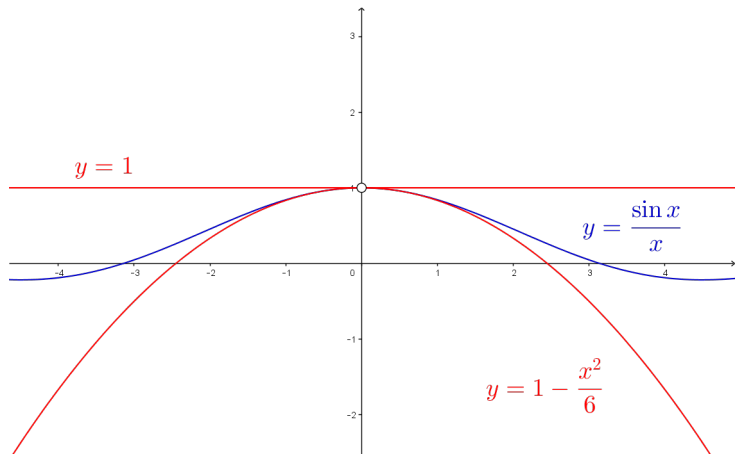
Theorem

Let k be a positive integer.

1. $\displaystyle\lim_{x \to +\infty} \frac{x^k}{e^x} = 0$

2. $\displaystyle\lim_{x \to +\infty} \frac{(\ln x)^k}{x} = 0$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Proof.

1 For any $x > 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots > \frac{x^{k+1}}{(k+1)!}$$

and thus

$$0 < \frac{x^k}{e^x} < \frac{(k+1)!}{x}.$$

Moreover $\lim_{x \to +\infty} \frac{(k+1)!}{x} = 0$. Therefore

$$\lim_{x \to +\infty} \frac{x^k}{e^x} = 0.$$

2 Let $x = e^y$. Then $x \to +\infty$ as $y \to +\infty$ and $\ln x = y$. We have

$$\lim_{x \to +\infty} \frac{(\ln x)^k}{x} = \lim_{y \to +\infty} \frac{y^k}{e^y} = 0.$$

$\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

1. $\displaystyle \lim_{x \to 4} \frac{x^2 - 16}{\sqrt{x} - 2}$
$$= \lim_{x \to 4} \frac{(x-4)(x+4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}$$
$$= \lim_{x \to 4} \frac{(x-4)(x+4)(\sqrt{x}+2)}{x-4}$$
$$= \lim_{x \to 4} (x+4)(\sqrt{x}+2) = 32$$

2. $\displaystyle \lim_{x \to +\infty} \frac{3e^{2x} + e^x - x^4}{4e^{2x} - 5e^x + 2x^4}$
$$= \lim_{x \to +\infty} \frac{3 + e^{-x} - x^4 e^{-2x}}{4 - 5e^{-x} + 2x^4 e^{-2x}} = \frac{3}{4}$$

3. $\displaystyle \lim_{x \to +\infty} \frac{\ln(2e^{4x} + x^3)}{\ln(3e^{2x} + 4x^5)}$
$$= \lim_{x \to +\infty} \frac{4x + \ln(2 + x^3 e^{-4x})}{2x + \ln(3 + 4x^5 e^{-2x})}$$
$$= \lim_{x \to +\infty} \frac{4 + \frac{\ln(2 + x^3 e^{-4x})}{x}}{2 + \frac{\ln(3 + 4x^5 e^{-2x})}{x}} = 2$$

4. $\displaystyle \lim_{x \to -\infty} (x + \sqrt{x^2 - 2x})$
$$= \lim_{x \to -\infty} \frac{(x + \sqrt{x^2 - 2x})(x - \sqrt{x^2 - 2x})}{x - \sqrt{x^2 - 2x}}$$
$$= \lim_{x \to -\infty} \frac{2x}{x - \sqrt{x^2 - 2x}}$$
$$= \lim_{x \to -\infty} \frac{2}{1 + \sqrt{1 - \frac{2}{x}}} = 1$$

Limits   Sequences
Differentiation   **Limits of functions**
Integration   Continuity of functions

Example

5. $\displaystyle \lim_{x \to 0} \frac{\sin 6x - \sin x}{\sin 4x - \sin 3x}$ $=$ $\displaystyle \lim_{x \to 0} \frac{\frac{6 \sin 6x}{6x} - \frac{\sin x}{x}}{\frac{4 \sin 4x}{4x} - \frac{3 \sin 3x}{3x}} = \frac{6 - 1}{4 - 3} = 5$

6. $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x \tan x}$ $=$ $\displaystyle \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x \frac{\sin x}{\cos x}(1 + \cos x)}$

$=$ $\displaystyle \lim_{x \to 0} \frac{(1 - \cos^2 x) \cos x}{x \sin x (1 + \cos x)}$

$=$ $\displaystyle \lim_{x \to 0} \left(\frac{\sin x}{x} \right) \frac{\cos x}{1 + \cos x} = \frac{1}{2}$

7. $\displaystyle \lim_{x \to 0} \frac{e^{2x} - 1}{\ln(1 + 3x)}$ $=$ $\displaystyle \lim_{x \to 0} \frac{2}{3} \cdot \frac{e^{2x} - 1}{2x} \cdot \frac{3x}{\ln(1 + 3x)} = \frac{2}{3}$

8. $\displaystyle \lim_{x \to 0} \frac{x \ln(1 + \sin x)}{1 - \sqrt{\cos x}}$ $=$ $\displaystyle \lim_{x \to 0} \frac{x(1 + \sqrt{\cos x})(1 + \cos x) \ln(1 + \sin x)}{1 - \cos^2 x}$

$=$ $\displaystyle \lim_{x \to 0} \frac{x}{\sin x} \cdot \frac{\ln(1 + \sin x)}{\sin x}(1 + \sqrt{\cos x})(1 + \cos x)$

$=$ 4

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let $g(u)$ be a function of u and $u = f(x)$ be a function of x.
Suppose

1. $\lim\limits_{x \to a} f(x) = b \in [-\infty, +\infty]$

2. $\lim\limits_{u \to b} g(u) = l$

3. $f(x) \neq b$ when $x \neq a$ or $g(b) = l$.

Then

$$\lim_{x \to a} (g \circ f)(x) = l.$$

$$x \xrightarrow{\ f\ } u = f(x) \xrightarrow{\ g\ } (g \circ f)(x) = g(u) = g(f(x))$$

Limits   Sequences
Differentiation   **Limits of functions**
Integration   Continuity of functions

Example

1. $\displaystyle\lim_{x\to 0} \frac{e^{4x^3}-1}{x^2 \sin 3x}$ $\quad = \quad \displaystyle\lim_{x\to 0} \frac{4}{3}\left(\frac{3x}{\sin 3x}\right)\left(\frac{e^{4x^3}-1}{4x^3}\right)$

$\quad = \quad \dfrac{4}{3}\displaystyle\lim_{y\to 0}\left(\frac{e^y-1}{y}\right) \quad (y = 4x^3)$

$\quad = \quad \dfrac{4}{3}$

2. $\displaystyle\lim_{x\to 0} \frac{\ln(1+2\tan x)}{x}$ $\quad = \quad \displaystyle\lim_{x\to 0}\left(\frac{2}{\cos x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{\ln(1+2\tan x)}{2\tan x}\right)$

$\quad = \quad 2\displaystyle\lim_{y\to 0}\left(\frac{\ln(1+y)}{y}\right) \quad (y = 2\tan x)$

$\quad = \quad 2$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition (Continuity)

Let $f(x)$ be a real valued function. We say that $f(x)$ is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

In other words, $f(x)$ is continuous at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

We say that $f(x)$ is continuous on an interval in $\mathbb{R}$ if $f(x)$ is continuous at every point on the interval.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Let $g(u)$ be a function in u and $u = f(x)$ be a function in x.
Suppose $g(u)$ is continuous and the limit of $f(x)$ at $x = a$ exists.
Then
$$\lim_{x \to a} (g \circ f)(x) = \lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

$$x \xrightarrow{f} u = f(x) \xrightarrow{g} (g \circ f)(x) = g(u) = g(f(x))$$

Limits   Sequences
Differentiation   Limits of functions
Integration   **Continuity of functions**

Theorem

1. *For any non-negative integer n, $f(x) = x^n$ is continuous on $\mathbb{R}$.*

2. *The functions $e^x, \cos x, \sin x$ are continuous on $\mathbb{R}$.*

3. *The logarithmic function $\ln x$ is continuous on $\mathbb{R}^+$.*

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Theorem

Suppose $f(x)$, $g(x)$ are continuous functions and c is a real number. Then the following functions are continuous.

1. $f(x) + g(x)$

2. $cf(x)$

3. $f(x)g(x)$

4. $\dfrac{f(x)}{g(x)}$ *at the points where $g(x) \neq 0$.*

5. $(f \circ g)(x)$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Definition

The **absolute value** of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

$y = |x|$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
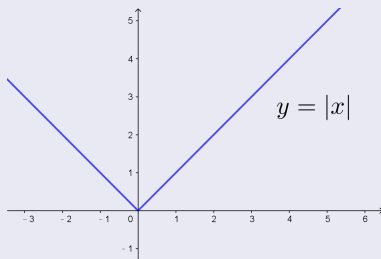
Example (Piecewise defined function)

$$y = f(x) = \begin{cases} x^2 + 2, & \text{if } x \leq 1 \\ \sqrt{x-1}, & \text{if } 1 < x < 5 \\ 1, & \text{if } x = 5 \\ 7 - x, & \text{if } x > 5 \end{cases}$$
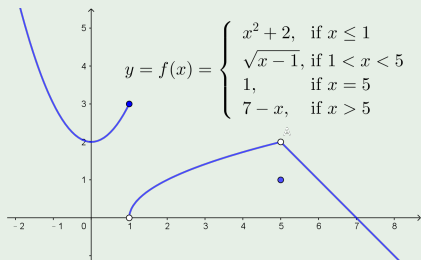
a	1	5
$\displaystyle\lim_{x \to a^-} f(x)$	3	2
$\displaystyle\lim_{x \to a^+} f(x)$	0	2
$\displaystyle\lim_{x \to a} f(x)$	does not exist	2

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

$$y = ||x| - 1| = \begin{cases} -x - 1, \text{ if } x < -1 \\ x + 1, \quad \text{ if } -1 \le x < 0 \\ -x + 1, \text{ if } 0 \le x < 1 \\ x - 1, \quad \text{ if } x \ge 1 \end{cases}$$

Limits   Sequences
Differentiation   Limits of functions
Integration   **Continuity of functions**
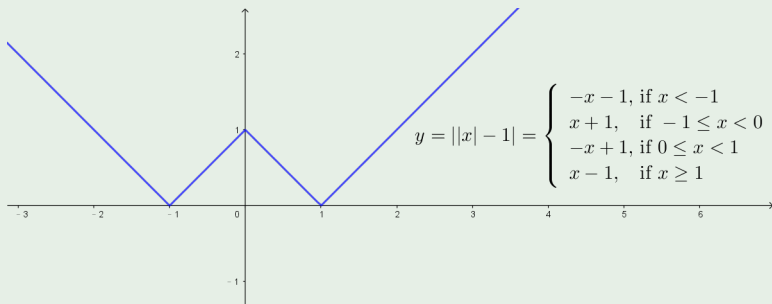
Theorem

A function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

The theorem is usually used to check whether a piecewise defined function is continuous.

Limits   Sequences
Differentiation   Limits of functions
Integration   **Continuity of functions**

The following functions are not continuous at $x = a$.
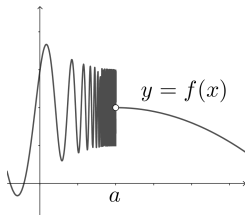
$$\lim_{x \to a^-} f(x) \text{ does not exist}$$
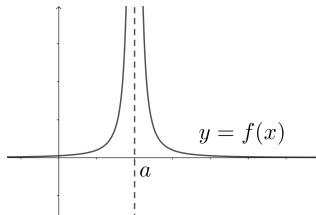
$$\lim_{x \to a} f(x) \text{ does not exist}$$

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

$$\lim_{x \to a} f(x) \neq f(a)$$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
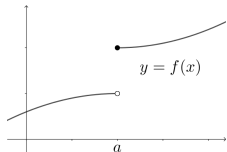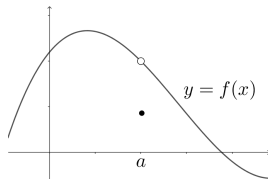
Example

Given that the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ a & \text{if } x = 2 \\ x^2 + b & \text{if } x > 2 \end{cases}$$

is continuous at $x = 2$. Find the value of a and b.

Solution

Note that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1) = 3$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + b) = 4 + b$$
$$f(2) = a$$

Since $f(x)$ is continuous at $x = 2$, we have $3 = 4 + b = a$ which implies $a = 3$ and $b = -1$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

Example

Prove that the function

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Proof.

Let $x_n = \dfrac{2}{(2n+1)\pi}$ for $n = 1, 2, 3, \ldots$. Then $\lim\limits_{n\to\infty} x_n = 0$ and

$$f(x_n) = \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n.$$

Thus $\lim\limits_{n\to\infty} f(x_n)$ does not exist. Therefore $f(x)$ is not continuous at $x = 0$. $\qquad\square$

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions

$$y = f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$f(x)$ is not continuous at $x = 0$.

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
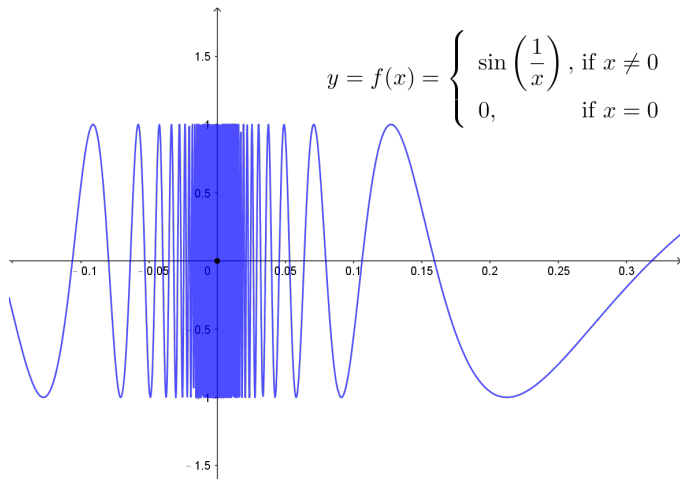
Theorem (Intermediate value theorem)

*Suppose $f(x)$ is a function which is **continuous** on $[a, b]$. Then for any real number η between $f(a)$ and $f(b)$, there exists $\xi \in (a, b)$ such that $f(\xi) = \eta$.*

Limits
Differentiation
Integration

Sequences
Limits of functions
Continuity of functions
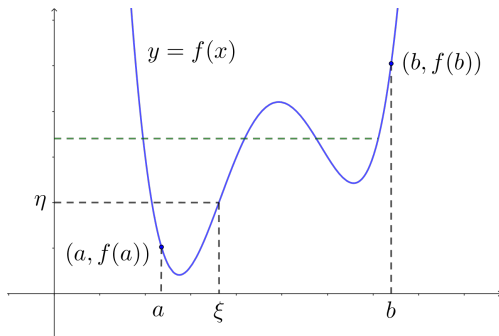
Theorem (Extreme value theorem)

*Suppose $f(x)$ is a function which is **continuous** on a **closed and bounded** interval $[a, b]$. Then there exists $\alpha, \beta \in [a, b]$ such that*

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
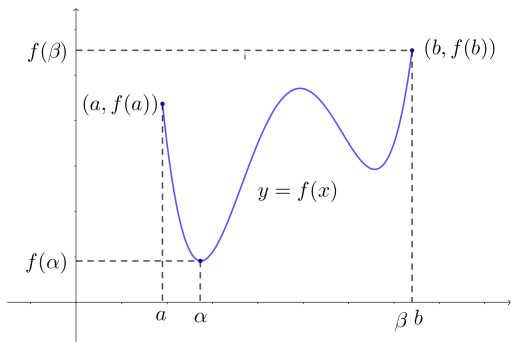
Differentiable functions

Definition (Differentiable function)

Let $f(x)$ be a function. Denote

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and we say that $f(x)$ is **differentiable** at $x = a$ if the above limit exists. We say that $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable at every point in (a, b).

The above limit can also be written as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation
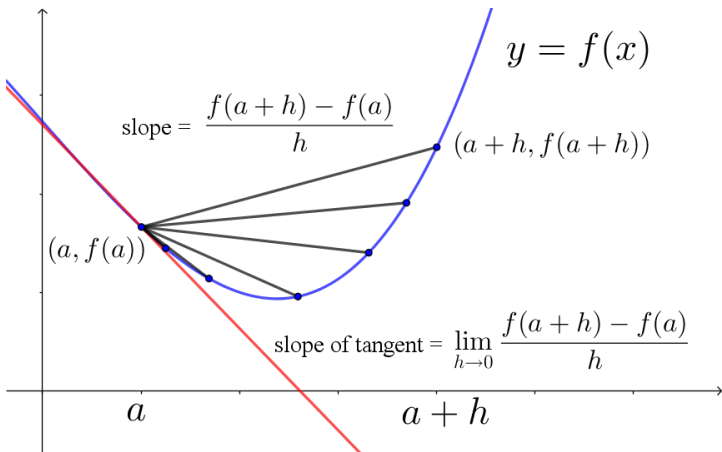
Figure: Definition of derivative

Limits
Derivatives
Differentiation
Mean value theorem
Integration
Application of Differentiation

slope of tangent at $x = 2$ is -1

$y = f(x)$

$y = f'(x)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
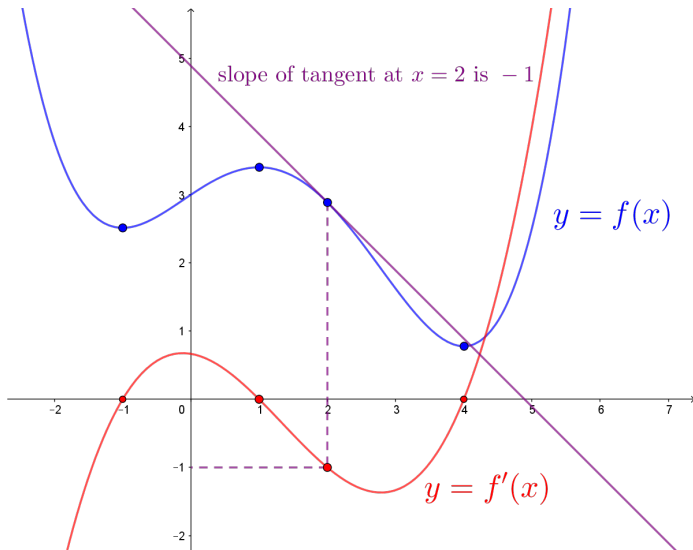
Theorem

If $f(x)$ differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Differentiable at $x = a$ ⟹ Continuous at $x = a$

Proof.

Suppose $f(x)$ is differentiable at $x = a$. Then

$$
\begin{aligned}
\lim_{x \to a} (f(x) - f(a)) &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) (x - a) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) \lim_{x \to a} (x - a) \\
&= f'(a) \cdot 0 = 0
\end{aligned}
$$

Therefore $f(x)$ is continuous at $x = a$. □

Note that the converse of the above theorem does not hold. The function
$f(x) = |x|$ is continuous but not differentiable at 0.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

The following functions are not differentiable at $x = a$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
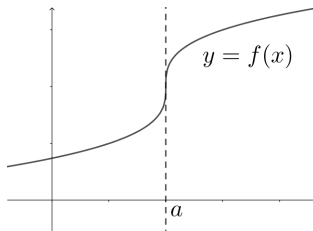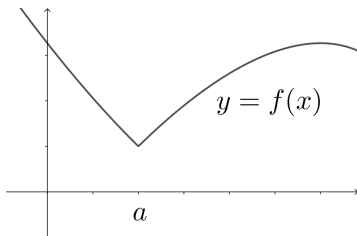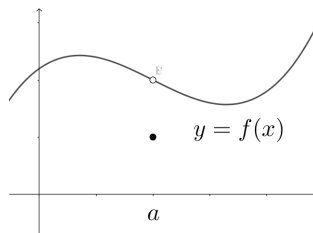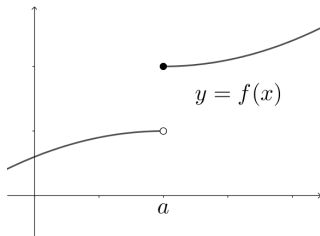
Example

1. $f(x) = e^x$: $f'(0) = \lim\limits_{h \to 0} \dfrac{e^h - e^0}{h} = \lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

2. $f(x) = \ln x$: $f'(1) = \lim\limits_{h \to 0} \dfrac{\ln(1 + h) - \ln 1}{h} = \lim\limits_{h \to 0} \dfrac{\ln(1 + h)}{h} = 1$.

3. $f(x) = \sin x$: $f'(0) = \lim\limits_{h \to 0} \dfrac{\sin h - \sin 0}{h} = \lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find the values of a, b if $f(x) = \begin{cases} 4x - 1, & \text{if } x \leq 1 \\ ax^2 + bx, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Solution: Since $f(x)$ is differentiable at $x = 1$, $f(x)$ is continuous at $x = 1$ and we have

$$\lim_{x \to 1^+} f(x) = f(1) \Rightarrow \lim_{x \to 1^+} (ax^2 + bx) = a + b = 3.$$

Moreover, $f(x)$ is differentiable at $x = 1$ and we have

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(4(1+h) - 1) - 3}{h} = 4$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{a(1+h)^2 - b(1+h) - 3}{h} = 2a + b$$

Therefore $\begin{cases} a + b = 3 \\ 2a + b = 4 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Definition (First derivative)

Let $y = f(x)$ be a differentiable function on (a, b). The **first derivative** of $f(x)$ is the function on (a, b) defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Theorem

Let $f(x)$ and $g(x)$ be differentiable functions and c be a real number. Then

1. $(f + g)'(x) = f'(x) + g'(x)$
2. $(cf)'(x) = cf'(x)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

1. $\dfrac{d}{dx} x^n = nx^{n-1}$, $n \in \mathbb{Z}^+$, *for* $x \in \mathbb{R}$

2. $\dfrac{d}{dx} e^x = e^x$ *for* $x \in \mathbb{R}$

3. $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ *for* $x > 0$

4. $\dfrac{d}{dx} \cos x = -\sin x$ *for* $x \in \mathbb{R}$

5. $\dfrac{d}{dx} \sin x = \cos x$ *for* $x \in \mathbb{R}$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof ($\dfrac{d}{dx}x^n = nx^{n-1}$)

Let $y = x^n$. For any $x \in \mathbb{R}$, we have

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{(x+h-x)((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1})}{h} \\
&= \lim_{h \to 0} ((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}) \\
&= nx^{n-1}
\end{aligned}
$$

Note that the above proof is valid only when $n \in \mathbb{Z}^+$ is a positive integer.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof ($\frac{d}{dx}e^x = e^x$)

Let $y = e^x$. For any $x \in \mathbb{R}$, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x.$$

(Alternative proof)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \\
&= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= e^x
\end{aligned}
$$

In general, differentiation cannot be applied term by term to infinite series. The second proof is valid only after we prove that this can be done to **power series**.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof

$\left(\dfrac{d}{dx} \ln x = \dfrac{1}{x} \right)$ Let $f(x) = \ln x$. For any $x > 0$, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\ln(x + h) - \ln x}{h} = \lim_{h \to 0} \frac{\ln \left(1 + \frac{h}{x} \right)}{h} = \frac{1}{x}.$$

$\left(\dfrac{d}{dx} \cos x = -\sin x \right)$ Let $f(x) = \cos x$. For any $x \in \mathbb{R}$, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = \lim_{h \to 0} \frac{-2 \sin \left(x + \frac{h}{2} \right) \sin \left(\frac{h}{2} \right)}{h} = -\sin x.$$

$\left(\dfrac{d}{dx} \sin x = \cos x \right)$ Let $f(x) = \sin x$. For any $x \in \mathbb{R}$, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{2 \cos \left(x + \frac{h}{2} \right) \sin \left(\frac{h}{2} \right)}{h} = \cos x.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Definition

Let $a > 0$ be a positive real number. For $x \in \mathbb{R}$, we define

$$a^x = e^{x \ln a}.$$

Theorem

Let $a > 0$ be a positive real number. We have

1. $a^{x+y} = a^x a^y$ *for any $x, y \in \mathbb{R}$*

2. $\dfrac{d}{dx} a^x = a^x \ln a.$

Proof.

1. $a^{x+y} = e^{(x+y) \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$

2. $\dfrac{d}{dx} a^x = \dfrac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$

$\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Show that $f(x)$ is not differentiable at $x = 0$.

Proof.

Observe that

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Thus the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$. □

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
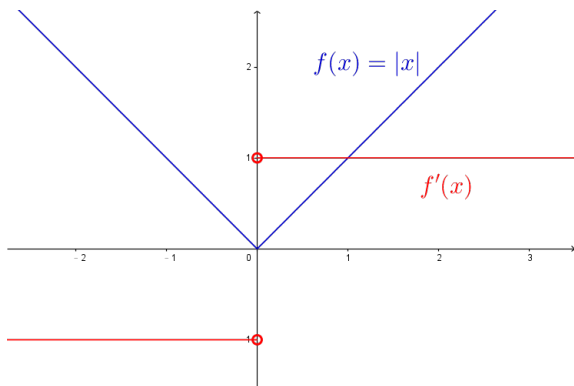
Figure: $f(x) = |x|$ is not differentiable at $x = 0$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Exercise (True or False)

Suppose $f(x)$ is bounded and is differentiable on (a, b). Then

1. $f'(x)$ *is differentiable on* (a, b).
   Answer: **F**

2. $f'(x)$ *is continuous on* (a, b).
   Answer: **F**

3. $f'(x)$ *is bounded on* (a, b).
   Answer: **F**

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Let $f(x) = |x|x$ for $x \in \mathbb{R}$. Find $f'(x)$.

Solution: When $x < 0$, $f(x) = -x^2$ and $f'(x) = -2x$. When $x > 0$, $f(x) = x^2$ and $f'(x) = 2x$. When $x = 0$, we have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|h - 0}{h} = \lim_{h \to 0} |h| = 0$$

Thus $f'(0) = 0$. Therefore

$$
\begin{aligned}
f'(x) &= \begin{cases}
-2x, & \text{if } x < 0 \\
0, & \text{if } x = 0 \\
2x, & \text{if } x > 0
\end{cases} \\
&= 2|x|.
\end{aligned}
$$

Note that $f'(x) = 2|x|$ is continuous at $x = 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

$f'(x) = 2|x|$

$f(x) = |x|x$
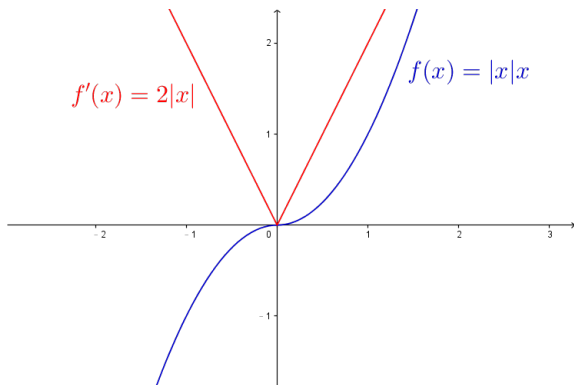
- $f(x)$ is differentiable at $x = 0$. ($f(x)$ is differentiable on $\mathbb{R}$.)
- $f'(x)$ is continuous on $\mathbb{R}$.
- $f'(x)$ is not differentiable at $x = 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

1. Find $f'(x)$ for $x \neq 0$.

2. Determine whether $f(x)$ is differentiable at $x = 0$.

Solution

1. When $x \neq 0$,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

2. We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) - \dfrac{1}{x}\cos\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ \text{undefined}, & \text{if } x = 0 \end{cases}$$

- $f(x)$ is not differentiable at $x = 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
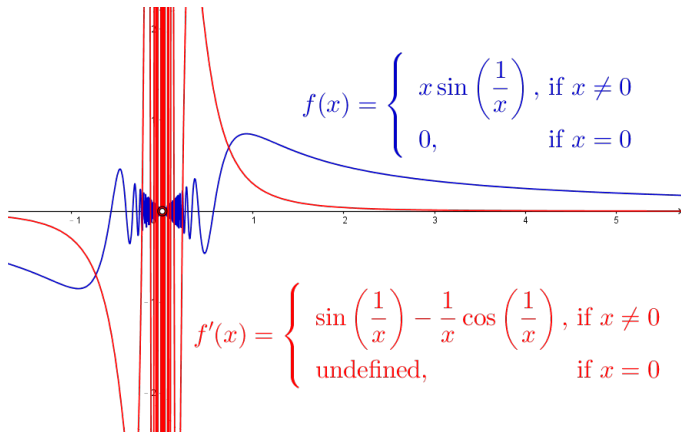
Example

Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

1. Find $f'(x)$.

2. Determine whether $f'(x)$ is continuous at $x = 0$.

Solution

1. When $x \neq 0$, we have

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left(-\frac{1}{x^2} \cos \frac{1}{x} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Solution

2. When $x = 0$, we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h}.$$

Since $\lim_{h \to 0} h = 0$ and $|\sin \frac{1}{h}| \leq 1$ is bounded, we have $f'(0) = 0$. Therefore

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Observe that

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist. We conclude that $f'(x)$ is not continuous at $x = 0$.
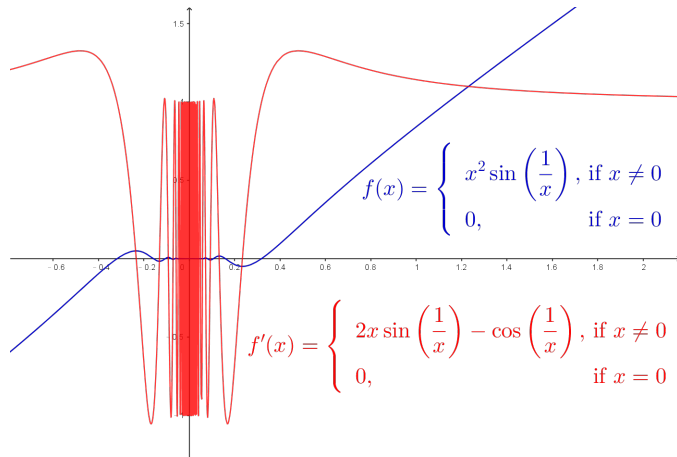
Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- $f'(0) = 0$ ($f(x)$ is differentiable on $\mathbb{R}$)
- $f'(x)$ is not continuous at $x = 0$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x^2}\right), \text{ if } x \neq 0 \\ 0, \qquad\qquad \text{ if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x^2}\right) - \dfrac{2}{x^3} \cos\left(\dfrac{1}{x^2}\right), \text{ if } x \neq 0 \\ 0, \qquad\qquad\qquad\qquad\qquad \text{ if } x = 0 \end{cases}$$
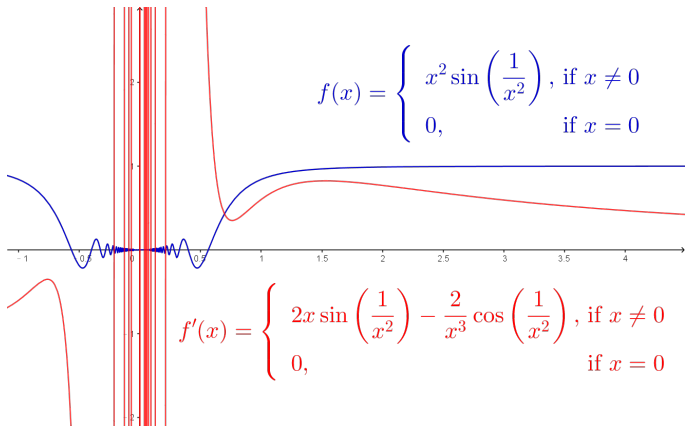
- $f'(0) = 0$ ($f(x)$ is differentiable on $\mathbb{R}$)
- $f'(x)$ is not continuous at $x = 0$
- $f'(x)$ is not bounded near $x = 0$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

$f(x)$	$f(x)$ is continuous at $x = 0$	$f(x)$ is differentiable at $x = 0$	$f'(x)$ is continuous at $x = 0$
$|x|$	Yes	No	Not applicable
$|x|x$	Yes	Yes	Yes
$x \sin \left(\dfrac{1}{x} \right); \; f(0) = 0$	Yes	No	Not applicable
$x^2 \sin \left(\dfrac{1}{x} \right); \; f(0) = 0$	Yes	Yes	No

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

The following diagram shows the relations between the existence of limit, continuity and differentiability of a function at a point a. (Examples in the bracket is for $a = 0$.)

$$\textbf{Second differentiable} \qquad \left(f(x) = \frac{\sin x}{x};\ f(0) = 1\right)$$
$$\Downarrow$$
$$\textbf{Continuously differentiable} \qquad \left(f(x) = |x|x\right)$$
$$\Downarrow$$
$$\textbf{Differentiable} \qquad \left(f(x) = x^2 \sin(x^{-1});\ f(0) = 0\right)$$
$$\Downarrow$$
$$\textbf{Continuous} \qquad \left(f(x) = |x|\right)$$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation
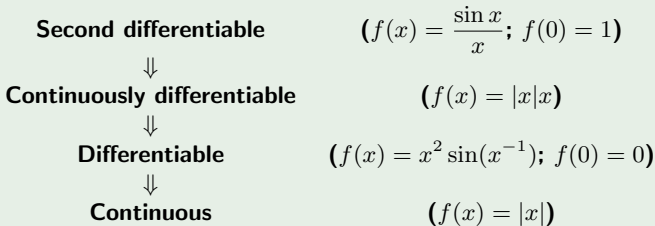
Rules of differentiation

Theorem (Basic formulas for differentiation)

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem (Product rule and quotient rule)

Let u and v be differentiable functions of x. Then

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Proof

Let $u = f(x)$ and $v = g(x)$.

$$
\begin{aligned}
\frac{d}{dx}uv &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \left(\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right) \\
&= \lim_{h \to 0} \left(f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right) \\
&= u\frac{dv}{dx} + v\frac{du}{dx}
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof.

$$
\begin{aligned}
\frac{d}{dx}\frac{u}{v} &= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h \to 0} \left(\frac{f(x+h)g(x) - f(x)g(x)}{hg(x)g(x+h)} - \frac{f(x)g(x+h) - f(x)g(x)}{hg(x)g(x+h)} \right) \\
&= \lim_{h \to 0} \left(g(x) \cdot \frac{f(x+h) - f(x)}{hg(x)g(x+h)} - f(x) \cdot \frac{g(x+h) - g(x)}{hg(x)g(x+h)} \right) \\
&= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}
\end{aligned}
$$

$\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem (Chain rule)

Let $y = f(u)$ be a function of u and $u = g(x)$ be a function of x. Suppose $g(x)$ is differentiable at $x = a$ and $f(u)$ is differentiation at $u = g(a)$. Then $f \circ g(x) = f(g(x))$ is differentiable at $x = a$ and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

In other words,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Limits
Derivatives
Differentiation
Mean value theorem
Integration
Application of Differentiation

Proof

$$(f \circ g)'(a)$$

$$= \lim_{h \to 0} \frac{f(g(a + h)) - f(g(a))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(a + h)) - f(g(a))}{g(a + h) - g(a)} \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

$$= \lim_{k \to 0} \frac{f(g(a) + k) - f(g(a))}{k} \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

(*Note that* $g(a + h) - g(a) = k \to 0$ *as* $h \to 0$ *because* $g(x)$ *is continuous.*)

$$= f'(g(a))g'(a)$$

The above proof is valid only if $g(a + h) - g(a) \neq 0$ whenever h is sufficiently close to 0. This is true when $g'(a) \neq 0$ because of the following proposition.

Proposition

Suppose $g(x)$ *is a function such that* $g'(a) \neq 0$. *Then there exists* $\delta > 0$ *such that if* $0 < |h| < \delta$, *then*

$$g(a + h) - g(a) \neq 0.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

When $g'(a) = 0$, we need another proposition.

Proposition

Suppose $f(u)$ is a function which is differentiable at $u = b$. Then there exists $\delta > 0$ and $M > 0$ such that

$$|f(b + h) - f(b)| < M|h| \text{ for any } |h| < \delta.$$

The proof of chain rule when $g'(a) = 0$ goes as follows. There exists $\delta > 0$ such that

$$|f(g(a + h)) - f(g(a))| < M|g(a + h) - g(a)| \text{ for any } |h| < \delta.$$

Therefore

$$\lim_{h \to 0} \left| \frac{f(g(a + h)) - f(g(a))}{h} \right| \leq \lim_{h \to 0} M \left| \frac{g(a + h) - g(a)}{h} \right| = 0$$

which implies $(f \circ g)'(a) = 0$.

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation

Example

The chain rule is used in the following way. Suppose u is a differentiable function of x. Then

$$
\begin{aligned}
\frac{d}{dx}u^n &= nu^{n-1}\frac{du}{dx} \\
\frac{d}{dx}e^u &= e^u\frac{du}{dx} \\
\frac{d}{dx}\ln u &= \frac{1}{u}\frac{du}{dx} \\
\frac{d}{dx}\cos u &= -\sin u\frac{du}{dx} \\
\frac{d}{dx}\sin u &= \cos u\frac{du}{dx}
\end{aligned}
$$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation

Example

1. $\dfrac{d}{dx}\sin^3 x \quad = \quad 3\sin^2 x \dfrac{d}{dx}\sin x = 3\sin^2 x \cos x$

2. $\dfrac{d}{dx}e^{\sqrt{x}} \quad = \quad e^{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

3. $\dfrac{d}{dx}\dfrac{1}{(\ln x)^2} \quad = \quad -\dfrac{2}{(\ln x)^3}\dfrac{d}{dx}\ln x = -\dfrac{2}{x(\ln x)^3}$

4. $\dfrac{d}{dx}\ln\cos 2x \quad = \quad \dfrac{1}{\cos 2x}(-\sin 2x)\cdot 2 = -\dfrac{2\sin 2x}{\cos 2x} = -2\tan 2x$

5. $\dfrac{d}{dx}\tan\sqrt{1+x^2} \quad = \quad \sec^2\sqrt{1+x^2}\cdot\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x = \dfrac{x\sec^2\sqrt{1+x^2}}{\sqrt{1+x^2}}$

6. $\dfrac{d}{dx}\sec^3\sqrt{\sin x} \quad = \quad 3\sec^2\sqrt{\sin x}(\sec\sqrt{\sin x}\tan\sqrt{\sin x})\dfrac{1}{2\sqrt{\sin x}}\cdot\cos x$

   $\qquad\qquad\qquad = \quad \dfrac{3\sec^3\sqrt{\sin x}\tan\sqrt{\sin x}\cos x}{2\sqrt{\sin x}}$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

7. $\dfrac{d}{dx} \cos^4 x \sin x$ $\quad = \quad \cos^4 x \cos x + 4 \cos^3 x (-\sin x) \sin x$

$\quad = \quad \cos^5 x - 4 \cos^3 x \sin^2 x$

8. $\dfrac{d}{dx} \dfrac{\sec 2x}{\ln x}$ $\quad = \quad \dfrac{\ln x (2 \sec 2x \tan 2x) - \sec 2x (\frac{1}{x})}{(\ln x)^2}$

$\quad = \quad \dfrac{\sec 2x (2x \tan 2x \ln x - 1)}{x (\ln x)^2}$

9. $e^{\frac{\tan x}{x}}$ $\quad = \quad e^{\frac{\tan x}{x}} \left(\dfrac{x \sec^2 x - \tan x}{x^2} \right)$

10. $\sin \left(\dfrac{\ln x}{\sqrt{1+x^2}} \right)$ $\quad = \quad \cos \left(\dfrac{\ln x}{\sqrt{1+x^2}} \right) \left(\dfrac{\sqrt{1+x^2}(\frac{1}{x}) - \ln x (\frac{2x}{2\sqrt{1+x^2}})}{1+x^2} \right)$

$\quad = \quad \left(\dfrac{1 + x^2 - x^2 \ln x}{x(1+x^2)^{\frac{3}{2}}} \right) \cos \left(\dfrac{\ln x}{\sqrt{1+x^2}} \right)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Definition (Implicit functions)

An **implicit function** is an equation of the form $F(x, y) = 0$. An implicit function may not define a function. Sometimes it defines a function when the domain and range are specified.

Theorem

Let $F(x, y) = 0$ be an implicit function. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

and we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Here $\frac{\partial F}{\partial x}$ is called the partial derivative of F with respect to x which is the derivative of F with respect to x while considering y as constant. Similarly the partial derivative $\frac{\partial F}{\partial y}$ is the derivative of F with respect to y while considering x as constant.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find $\dfrac{dy}{dx}$ for the following implicit functions.

1. $x^2 - xy - xy^2 = 0$
2. $\cos(xe^y) + x^2 \tan y = 1$

Solution

1. $2x - (y + xy') - (y^2 + 2xyy') = 0$

$xy' + 2xyy' = 2x - y - y^2$

$y' = \dfrac{2x - y - y^2}{x + 2xy}$

2. $-\sin(xe^y)(e^y + xe^yy') + 2x\tan y + x^2\sec^2 yy' = 0$

$x^2\sec^2 yy' - xe^y\sin(xe^y)y' = e^y\sin(xe^y) - 2x\tan y$

$y' = \dfrac{e^y\sin(xe^y) - 2x\tan y}{x^2\sec^2 y - xe^y\sin(xe^y)}$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Suppose $f(y)$ is a differentiable function with $f'(y) \neq 0$ for any y. Then the inverse function $y = f^{-1}(x)$ of $f(y)$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In other words,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Proof.

$$\begin{aligned}
f(f^{-1}(x)) &= x \\
f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}
\end{aligned}$$

$\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

1. For $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.$$

2. For $\cos^{-1} : [-1, 1] \to [0, \pi]$,

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

3. For $\tan^{-1} : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.$$

Proof.

1

$$
\begin{aligned}
y &= \sin^{-1} x \\
\sin y &= x \\
\cos y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1 - \sin^2 y}} \ \left(\text{Note: } \cos y \geq 0 \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right) \\
&= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}
$$

The other parts can be proved similarly. $\qquad\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find $\dfrac{dy}{dx}$ if $y = x^x$.

Solution

There are 2 methods.
Method 1. Note that $y = x^x = e^{x \ln x}$. Thus

$$\frac{dy}{dx} = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x).$$

Method 2. Taking logarithm on both sides, we have

$$
\begin{aligned}
\ln y &= x \ln x \\
\frac{1}{y}\frac{dy}{dx} &= 1 + \ln x \\
\frac{dy}{dx} &= y(1 + \ln x) \\
&= x^x(1 + \ln x)
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Let u and v be functions of x. Show that

$$\frac{d}{dx}u^v = u^v v' \ln u + u^{v-1} v u'.$$

Proof.

We have

$$
\begin{aligned}
\frac{d}{dx}u^v &= \frac{d}{dx}e^{v \ln u} \\
&= e^{v \ln u}\left(v' \ln u + v \cdot \frac{u'}{u}\right) \\
&= u^v\left(v' \ln u + \frac{vu'}{u}\right) \\
&= u^v v' \ln u + u^{v-1} v u'
\end{aligned}
$$

$\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Second and higher derivatives

Definition (Second and higher derivatives)

Let $y = f(x)$ be a function. The **second derivative** of $f(x)$ is the function

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The second derivative of $y = f(x)$ is also denoted as $f''(x)$ or y''. Let n be a non-negative integer. The n-**th derivative** of $y = f(x)$ is defined inductively by

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \text{ for } n \geq 1$$

$$\frac{d^0y}{dx^0} = y$$

The n-th derivative is also denoted as $f^{(n)}(x)$ or $y^{(n)}$. Note that $f^{(0)}(x) = f(x)$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find $\dfrac{d^2y}{dx^2}$ for the following functions.

1. $y = \ln(\sec x + \tan x)$
2. $x^2 - y^2 = 1$

Solution

1. $\begin{aligned}
y' &= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) \\
&= \sec x \\
y'' &= \sec x \tan x
\end{aligned}$

2. $\begin{aligned}
2x - 2yy' &= 0 \\
y' &= \frac{x}{y} \\
y'' &= \frac{y - xy'}{y^2} \\
&= \frac{y - x(\frac{x}{y})}{y^2} \\
&= \frac{y^2 - x^2}{y^3}
\end{aligned}$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem (Leinbiz's rule)

Let u and v be differentiable function of x. Then

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binormial coefficient.

Example

$$
\begin{aligned}
(uv)^{(0)} &= u^{(0)}v^{(0)} \\
(uv)^{(1)} &= u^{(1)}v^{(0)} + u^{(0)}v^{(1)} \\
(uv)^{(2)} &= u^{(2)}v^{(0)} + 2u^{(1)}v^{(1)} + u^{(0)}v^{(2)} \\
(uv)^{(3)} &= u^{(3)}v^{(0)} + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + u^{(0)}v^{(3)} \\
(uv)^{(4)} &= u^{(4)}v^{(0)} + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + u^{(0)}v^{(4)}
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof

*We prove the Leibniz's rule by induction on n. When $n = 0$,
$(uv)^{(0)} = uv = u^{(0)}v^{(0)}$. Assume that for some nonnegative m,*

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}.$$

Then

$$
\begin{aligned}
& (uv)^{(m+1)} \\
=\ & \frac{d}{dx}(uv)^{(m)} \\
=\ & \frac{d}{dx} \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)} \\
=\ & \sum_{k=0}^{m} \binom{m}{k} (u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)})
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof.

$$
\begin{aligned}
= & \sum_{k=0}^{m} \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k+1)} \\
= & \sum_{k=0}^{m} \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m-(k-1))} v^{(k)} \\
= & \sum_{k=0}^{m} \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m-k+1)} v^{(k)} \\
= & \sum_{k=0}^{m+1} \left(\binom{m}{k} + \binom{m}{k-1} \right) u^{(m-k+1)} v^{(k)} \\
= & \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(m+1-k)} v^{(k)}
\end{aligned}
$$

Here we use the convention $\binom{m}{-1} = \binom{m}{m+1} = 0$ in the second last equality. This completes the induction step and the proof of the Leibniz's rule. $\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Let $y = x^2 e^{3x}$. Find $y^{(n)}$ where n is a nonnegative integer.

Solution

Let $u = x^2$ and $v = e^{3x}$. Then $u^{(0)} = x^2$, $u^{(1)} = 2x$, $u^{(2)} = 2$ and $u^{(k)} = 0$ for $k \geq 3$. On the other hand, $v^{(k)} = 3^k e^{3x}$ for any $k \geq 0$. Therefore by Leibniz's rule, we have

$$
\begin{aligned}
y^{(n)} &= \binom{n}{0} u^{(0)} v^{(n)} + \binom{n}{1} u^{(1)} v^{(n-1)} + \binom{n}{2} u^{(2)} v^{(n-2)} \\
&= x^2 (3^n e^{3x}) + n(2x)(3^{n-1} e^{3x}) + \frac{n(n-1)}{2!}(2)(3^{n-2} e^{3x}) \\
&= (3^n x^2 + 2 \cdot 3^{n-1} nx + 3^{n-2}(n^2 - n)) e^{3x} \\
&= 3^{n-2}(9x^2 + 6nx + n^2 - n) e^{3x}
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Mean value theorem

Definition (Increasing and decreasing function)

Let $f(x)$ be a function. We say that $f(x)$ is

1. **monotonic increasing** (**monotonic decreasing**), or simply **increasing** (**decreasing**), if for any x, y with $x < y$, we have $f(x) \leq f(y)$ ($f(x) \geq f(y)$).

2. **strictly increasing** (**strictly decreasing**) if for any x, y with $x < y$, we have $f(x) < f(y)$ ($f(x) > f(y)$).

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Suppose $f(x)$ is a function which is differentiable on (a, b).

1. If $f(x)$ attains its maximum or minimum at $x = c \in (a, b)$, then
$f'(c) = 0$.
Answer: **T**

2. If $f'(c) = 0$, then $f(x)$ attains its maximum or minimum at
$x = c \in (a, b)$.
Answer: **F**

3. If $f'(x) = 0$ for any $x \in (a, b)$, then $f(x)$ is constant on (a, b).
Answer: **T**

4. If $f(x)$ is strictly increasing on (a, b), then $f'(x) > 0$ for any $x \in (a, b)$.
Answer: **F**

5. If $f'(x) > 0$ for any (a, b), then $f(x)$ is strictly increasing on (a, b).
Answer: **T**

6. If $f(x)$ is monotonic increasing on (a, b), then $f'(x) \geq 0$ for any
$x \in (a, b)$.
Answer: **T**

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Let f be a function on (a, b) and $c \in (a, b)$ such that

1 *f is differentiable at $x = c$, and*

2 *either $f(x) \leq f(c)$ for any $x \in (a, b)$, or $f(x) \geq f(c)$ for any $x \in (a, b)$.*

Then $f'(c) = 0$.

Proof.

Suppose $f(x) \leq f(c)$ for any $x \in (a, b)$. The proof for the other case is essentially the same. For any $h < 0$ with $a < c + h < c$, we have $f(c + h) - f(c) \leq 0$ and h is negative. Thus

$$f'(c) = \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} \geq 0$$

On the other hand, for any $h > 0$ with $c < c + h < b$, we have $f(c + h) - f(c) \leq 0$ and h is positive. Thus we have

$$f'(c) = \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \leq 0$$

Therefore $f'(c) = 0$. $\qquad \square$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation

Example

$f'(x) > 0$ for any x

$\Downarrow$

Strictly increasing

$\Downarrow$

Monotonic increasing $\quad\Leftrightarrow\quad f'(x) \geq 0$ for any x

$y = e^x$

$y = x^3$

$y = \begin{cases} 0, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
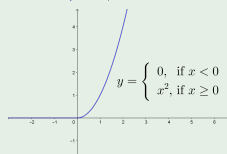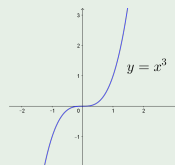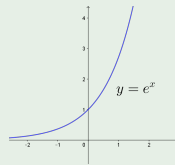
Theorem (Rolle's theorem)

Suppose $f(x)$ is a function which satisfies the following conditions.

1. *$f(x)$ is continuous on $[a, b]$.*

2. *$f(x)$ is differentiable on (a, b).*

3. *$f(a) = f(b)$*

Then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
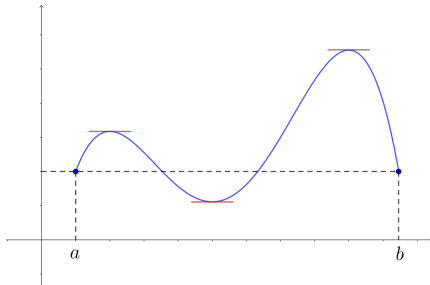
Proof.

By extreme value theorem, there exist $a \leq \alpha, \beta \leq b$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$

Since $f(a) = f(b)$, at least one of α, β can be chosen in (a, b) and we let it be ξ. Then we have $f'(\xi) = 0$ since $f(x)$ attains its maximum or minimum at ξ. $\qquad \square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem (Lagrange's mean value theorem)

Suppose $f(x)$ is a function which satisfies the following conditions.

1. *$f(x)$ is continuous on $[a, b]$.*

2. *$f(x)$ is differentiable on (a, b).*

Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
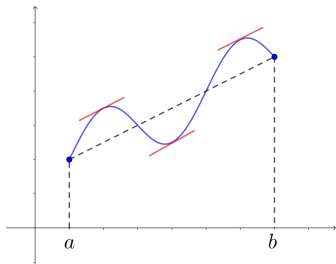
Proof.

Let $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$. Since $g(a) = g(b) = f(a)$,

by Rolle's theorem, there exists $\xi \in (a, b)$ such that

$$
\begin{aligned}
g'(\xi) &= 0 \\
f'(\xi) - \frac{f(b) - f(a)}{b - a} &= 0 \\
f'(\xi) &= \frac{f(b) - f(a)}{b - a}
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Let $f(x)$ be a function which is differentiable on (a, b). Then $f(x)$ is monotonic increasing if and only if $f'(x) \geq 0$ for any $x \in (a, b)$.

Proof. Suppose $f(x)$ is monotonic increasing on (a, b). Then for any $x \in (a, b)$, we have $f(x + h) - f(x) \geq 0$ for any $h > 0$ and thus

$$f'(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \geq 0.$$

On the other hand, suppose $f'(x) \geq 0$ for any $x \in (a, b)$. Then for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta$, by Lagrange's mean value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) \geq 0.$$

Therefore $f(x)$ is monotonic increasing on (a, b). □

Corollary

$f(x)$ is constant on (a, b) if and only if $f'(x) = 0$ for any $x \in (a, b)$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

If $f(x)$ is a differentiable function such that $f'(x) > 0$ for any $x \in (a, b)$, then $f(x)$ is strictly increasing.

Proof.

Suppose $f'(x) > 0$ for any $x \in (a, b)$. Then for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta$, by Lagrange's mean value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) > 0.$$

Therefore $f(x)$ is strictly increasing on (a, b). $\qquad\square$

The converse of the above theorem is false.

Example

$f(x) = x^3$ is strictly increasing on $\mathbb{R}$ but $f'(0) = 0$ is not positive.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Prove that $1 - \dfrac{1}{x} \le \ln x \le x - 1$ for any $x > 0$.

Solution. Let $f(x) = \ln x - \left(1 - \dfrac{1}{x}\right)$. Then $f'(x) = \dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x - 1}{x^2}$. Now $f'(1) = 0$ and

	$0 < x < 1$	$x > 1$
$f'(x)$	$-$	$+$

Therefore $f(x)$ attains its minimum at $x = 1$ and we have
$f(x) = \ln x - \dfrac{x - 1}{x} \ge f(1) = 0$ for any $x > 0$. On the other hand, let
$g(x) = x - 1 - \ln x$. Then $g'(x) = 1 - \dfrac{1}{x} = \dfrac{x - 1}{x}$. Now $g'(1) = 0$ and

	$0 < x < 1$	$x > 1$
$f'(x)$	$-$	$+$

Therefore $g(x)$ attains its minimum at $x = 1$ and we have
$g(x) = x - 1 - \ln x \ge g(1) = 0$ for any $x > 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Let $0 < \alpha < 1$. Prove that

$$1 + \alpha x - \frac{\alpha(1-\alpha)x^2}{2} < (1+x)^\alpha < 1 + \alpha x, \text{ for any } x > 0.$$

Solution. Let $f(x) = 1 + \alpha x - (1+x)^\alpha$. Then $f(0) = 0$ and for any $x > 0$,

$$f'(x) = \alpha - \frac{\alpha}{(1+x)^{1-\alpha}} > \alpha - \alpha = 0.$$

Therefore $f(x) > 0$ for any $x > 0$. On the other hand, let
$g(x) = (1+x)^\alpha - \left(1 + \alpha x - \frac{\alpha(1-\alpha)x^2}{2}\right)$. Then $g(0) = 0$ and for any $x > 0$,

$$
\begin{aligned}
g'(x) &= \frac{\alpha}{(1+x)^{1-\alpha}} - \alpha + \alpha(1-\alpha)x \\
&> \frac{\alpha}{1 + (1-\alpha)x} - \alpha(1 - (1-\alpha)x) \\
&= \frac{\alpha(1-\alpha)^2 x^2}{1 + (1-\alpha)x} > 0
\end{aligned}
$$

Therefore $g(x) > 0$ for any $x > 0$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem (Cauchy's mean value theorem)

Suppose $f(x)$ and $g(x)$ are functions which satisfies the following conditions.

1. $f(x), g(x)$ is continuous on $[a, b]$.
2. $f(x), g(x)$ is differentiable on (a, b).
3. $g'(x) \neq 0$ for any $x \in (a, b)$.

Then there exists $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let $h(x) = f(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$.

Since $h(a) = h(b) = f(a)$, by Rolle's theorem, there exists $\xi \in (a, b)$ such that

$$
\begin{aligned}
f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) &= 0 \\
\frac{f'(\xi)}{g'(\xi)} &= \frac{f(b) - f(a)}{g(b) - g(a)}
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

L'Hopital's rule

Theorem (L'Hopital's rule)

Let $a \in [-\infty, +\infty]$. Suppose f and g are differentiable functions such that

1. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ *(or $\pm\infty$).*
2. $g'(x) \neq 0$ *for any $x \neq a$ (on a neighborhood of a).*
3. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = L.$

Then the limit of $\dfrac{f(x)}{g(x)}$ at $x = a$ exists and $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = L.$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Proof.

We give here the proof for $a \in (-\infty, +\infty)$. For any $x \neq a$, by applying Cauchy's mean value theorem to $f(x)$, $g(x)$ on $[a, x]$ or $[x, a]$, there exists ξ between a and x such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Here we redefine $f(a) = g(a) = 0$, if necessary, so that f and g are continuous at a. Note that $\xi \to a$ as $x \to a$. We have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\xi)}{g'(\xi)} = L.$$

$\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example (Indeterminate form of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$)

1. $\displaystyle\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3}$ $=$ $\displaystyle\lim_{x\to 0}\frac{x\sin x}{3x^2} = \frac{1}{3}$

2. $\displaystyle\lim_{x\to 0}\frac{x^2}{\ln\sec x}$ $=$ $\displaystyle\lim_{x\to 0}\frac{2x}{\frac{\sec x\tan x}{\sec x}} = \lim_{x\to 0}\frac{2x}{\tan x} = \lim_{x\to 0}\frac{2}{\sec^2 x} = 2$

3. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x^3)}{x-\sin x}$ $=$ $\displaystyle\lim_{x\to 0}\frac{\frac{3x^2}{1+x^3}}{1-\cos x} = \lim_{x\to 0}\frac{3}{1+x^3}\lim_{x\to 0}\frac{x^2}{1-\cos x}$

   $=$ $\displaystyle 3\lim_{x\to 0}\frac{2x}{\sin x} = 6$

4. $\displaystyle\lim_{x\to +\infty}\frac{\ln(1+x^4)}{\ln(1+x^2)}$ $=$ $\displaystyle\lim_{x\to +\infty}\frac{\frac{4x^3}{1+x^4}}{\frac{2x}{1+x^2}} = \lim_{x\to +\infty}\frac{4x^3(1+x^2)}{2x(1+x^4)} = 2$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example (Indeterminate form of types $\infty - \infty$ and $0 \cdot \infty$)

$$
\begin{aligned}
5.\ \lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) &= \lim_{x \to 1} \frac{x - 1 - \ln x}{(x-1)\ln x} = \lim_{x \to 1} \frac{1 - \frac{1}{x}}{\frac{x-1}{x} + \ln x} \\
&= \lim_{x \to 1} \frac{x-1}{x-1+x\ln x} = \lim_{x \to 1} \frac{1}{2+\ln x} = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
6.\ \lim_{x \to 0} \cot 3x \tan^{-1} x &= \lim_{x \to 0} \frac{\tan^{-1} x}{\tan 3x} = \lim_{x \to 0} \frac{\frac{1}{1+x^2}}{3\sec^2 3x} \\
&= \lim_{x \to 0} \frac{1}{3(1+x^2)\sec^2 3x} = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
7.\ \lim_{x \to 0^+} x \ln \sin x &= \lim_{x \to 0^+} \frac{\ln \sin x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} \\
&= \lim_{x \to 0^+} \frac{-x^2 \cos x}{\sin x} = 0
\end{aligned}
$$

$$
\begin{aligned}
8.\ \lim_{x \to +\infty} x \ln \left(\frac{x+1}{x-1} \right) &= \lim_{x \to +\infty} \frac{\ln(x+1) - \ln(x-1)}{\frac{1}{x}} \\
&= \lim_{x \to +\infty} \frac{\frac{1}{x+1} - \frac{1}{x-1}}{-\frac{1}{x^2}} = \lim_{x \to +\infty} \frac{2x^2}{(x+1)(x-1)} = 2
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example (Indeterminate form of types 0^0, 1^∞ and ∞^0)

Evaluate the following limits.

1. $\lim\limits_{x \to 0^+} x^{\sin x}$

2. $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

3. $\lim\limits_{x \to +\infty} (1 + 2x)^{\frac{1}{3 \ln x}}$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Solution

1 $\ln\left(\lim_{x\to 0^+} x^{\sin x}\right) = \lim_{x\to 0^+} \ln(x^{\sin x}) = \lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\csc x}$

$= \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x\to 0^+} \dfrac{-\sin^2 x}{x \cos x} = 0.$

Thus $\lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$

2 $\ln\left(\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}\right) = \lim_{x\to 0} \ln(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} \dfrac{\ln \cos x}{x^2} = \lim_{x\to 0} \dfrac{-\tan x}{2x}$

$= \lim_{x\to 0} \dfrac{-\sec^2 x}{2} = -\dfrac{1}{2}.$

Thus $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}.$

3 $\ln\left(\lim_{x\to +\infty}(1+2x)^{\frac{3}{\ln x}}\right) = \lim_{x\to +\infty} \dfrac{3\ln(1+2x)}{\ln x} = \lim_{x\to +\infty} \dfrac{\frac{6}{1+2x}}{\frac{1}{x}} = 3.$

Thus $\lim_{x\to +\infty}(1+2x)^{\frac{1}{3\ln x}} = e^3.$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation

Example

The following shows some wrong use of L'Hopital rule.

1.

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sec x - 1}{e^{2x} - 1} &= \lim_{x \to 0} \frac{\sec x \tan x}{2e^{2x}} \\
&= \lim_{x \to 0} \frac{\sec^2 x \tan x + \sec^3 x}{4e^{2x}} \\
&= \frac{1}{4}
\end{aligned}
$$

This is wrong because $\lim_{x \to 0} e^{2x} \neq 0, \pm\infty$. One cannot apply

L'Hopital rule to $\lim_{x \to 0} \dfrac{\sec x \tan x}{2e^{2x}}$. The correct solution is

$$
\lim_{x \to 0} \frac{\sec x - 1}{e^{2x} - 1} = \lim_{x \to 0} \frac{\sec x \tan x}{2e^{2x}} = 0.
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

2.

$$\lim_{x \to +\infty} \frac{5x - 2\cos^2 x}{3x + \sin^2 x} = \lim_{x \to +\infty} \frac{5 + 2\cos x \sin x}{3 + \sin x \cos x}$$

$$= \lim_{x \to +\infty} \frac{2(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x}$$

$$= 2$$

This is wrong because $\lim\limits_{x \to +\infty} (5 + 2\cos x \sin x)$ and $\lim\limits_{x \to +\infty} (3 + \cos x \sin x)$ do not exist. One cannot apply L'Hopital rule to $\lim\limits_{x \to +\infty} \dfrac{5 + 2\cos x \sin x}{3 + \sin x \cos x}$. The correct solution is

$$\lim_{x \to +\infty} \frac{5x - 2\cos^2 x}{3x + \sin^2 x} = \lim_{x \to +\infty} \frac{5 - \frac{2\cos^2 x}{x}}{3 + \frac{\sin^2 x}{x}} = \frac{5}{3}.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Taylor series

Definition (Taylor polynomial)

Let $f(x)$ be a function such that the n-th derivative exists at $x = a$. The **Taylor polynomial** of degree n of $f(x)$ at $x = a$ is the polynomial

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem

The Taylor polynomial $p_n(x)$ of degree n of $f(x)$ at $x = a$ is the unique polynomial such that

$$p_n^{(k)}(a) = f^{(k)}(a) \text{ for } k = 0, 1, 2, \ldots, n.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find the Taylor polynomial $p_3(x)$ of degree 3 of $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$ at $x = 0$.

Solution. The derivatives $f^{(k)}(x)$ up to order 3 are

k	0	1	2	3
$f^{(k)}(x)$	$(1+x)^{\frac{1}{2}}$	$\dfrac{1}{2}(1+x)^{-\frac{1}{2}}$	$-\dfrac{1}{4}(1+x)^{-\frac{3}{2}}$	$\dfrac{3}{8}(1+x)^{-\frac{5}{2}}$
$f^{(k)}(0)$	1	$\dfrac{1}{2}$	$-\dfrac{1}{4}$	$\dfrac{3}{8}$

Therefore the Taylor polynomial of $f(x)$ of degree 3 at $x = 0$ is

$$
\begin{aligned}
p_3(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} \\
&= 1 + \left(\frac{1}{2}\right)x + \left(-\frac{1}{4}\right)\frac{x^2}{2!} + \left(\frac{3}{8}\right)\frac{x^3}{3!} \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}
\end{aligned}
$$

Figure: Taylor polynomials for $f(x) = \sqrt{1+x}$ at $x = 0$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
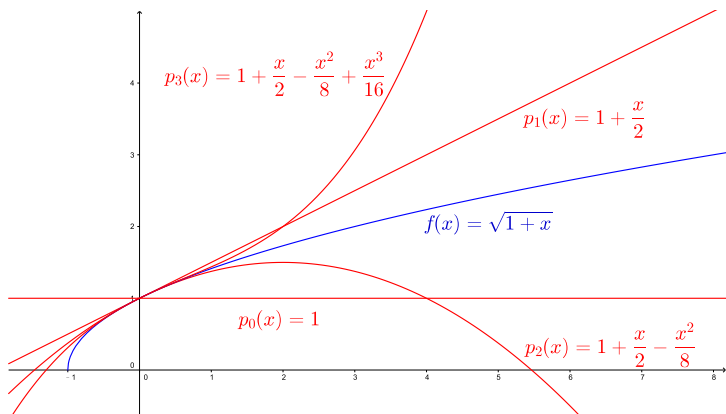
Example

Let $f(x) = \cos x$. The first few derivatives are

k	0	1	2	3	4
$f^{(k)}(x)$	$\cos x$	$-\sin x$	$-\cos x$	$-\sin x$	$\cos x$
$f^{(k)}(0)$	1	0	-1	0	1

We see that

$$f^{(n)}(x) = \begin{cases} (-1)^k \cos x, & \text{if } n = 2k \\ (-1)^k \sin x, & \text{if } n = 2k - 1 \end{cases} \text{ and } f^{(n)}(0) = \begin{cases} (-1)^k, & \text{if } n = 2k \\ 0, & \text{if } n = 2k - 1 \end{cases}$$

Therefore the Taylor polynomial of $f(x)$ of degree $n = 2k$ at $x = 0$ is

$$
\begin{aligned}
p_{2k}(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(2k)x^{2k}}(0)}{(2k)!} \\
&= 1 + (0)x + \frac{(-1)x^2}{2!} + \frac{(0)x^3}{3!} + \frac{(1)x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!}
\end{aligned}
$$
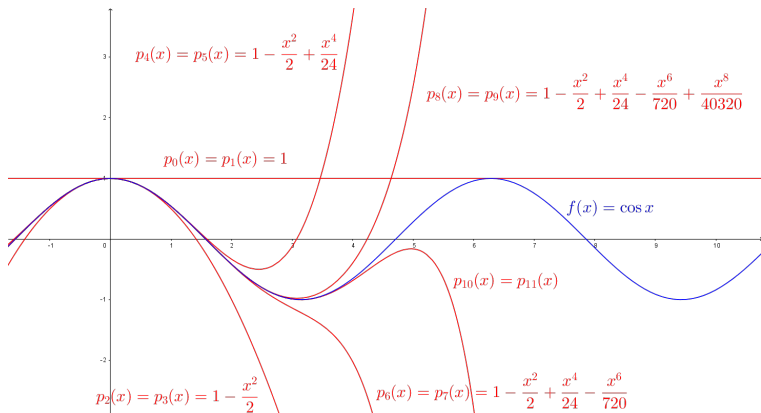
Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Figure: Taylor polynomials for $f(x) = \cos x$ at $x = 0$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find the Taylor polynomial of degree n of $f(x) = \dfrac{1}{x}$ at $x = 1$.

Solution. The derivatives $f^{(k)}(x)$ are

k	0	1	2	3	$\cdots$	n
$f^{(k)}(x)$	x^{-1}	$-x^{-2}$	$2x^{-3}$	$-6x^{-4}$	$\cdots$	$(-1)^n n! x^{-(n+1)}$
$f^{(k)}(1)$	1	-1	2	-6	$\cdots$	$(-1)^n n!$

Therefore the Taylor polynomial of $f(x)$ of degree n at $x = 1$ is

$$
\begin{aligned}
p_n(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \cdots + \frac{f^{(n)}(1)(x-1)^n}{n!} \\
&= 1 - (x-1) + \frac{2(x-1)^2}{2!} + \frac{(-6)(x-1)^3}{3!} + \cdots + \frac{(-1)^n n!(x-1)^n}{n!} \\
&= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n
\end{aligned}
$$

Figure: Taylor polynomials for $f(x) = \dfrac{1}{x}$ at $x = 1$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
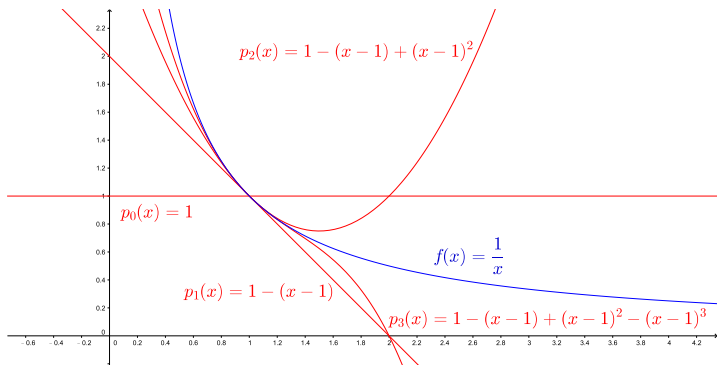
Example

Find the Taylor polynomial of $f(x) = (1+x)^\alpha$ at $x = 0$, where $\alpha \in \mathbb{R}$.
Solution. The derivatives are

$$
\begin{aligned}
f(x) &= (1+x)^\alpha \\
f'(x) &= \alpha(1+x)^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\
&\vdots \\
f^{(k)}(x) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}
\end{aligned}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Thus we have

$$
\begin{aligned}
f(0) &= 1 \\
f'(0) &= \alpha \\
f''(0) &= \alpha(\alpha - 1) \\
&\vdots \\
f^{(k)}(0) &= \alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - k + 1)
\end{aligned}
$$

Therefore the Taylor polynomial of $f(x) = (1 + x)^{\alpha}$ of degree n at $x = 0$ is

$$
\begin{aligned}
p_n(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} \\
&= 1 + \alpha x + \frac{\alpha(\alpha - 1)x^2}{2!} + \cdots + \frac{\alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - n + 1)x^n}{n!} \\
&= \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n
\end{aligned}
$$

where

$$
\binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - n + 1)}{n!}.
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

The Taylor polynomials of degree n for $f(x)$ at $x = 0$.

$f(x)$	Taylor polynomial
e^x	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots + \dfrac{(-1)^k x^{2k}}{(2k)!},\ n = 2k$
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots + \dfrac{(-1)^k x^{2k+1}}{(2k+1)!},\ n = 2k+1$
$\ln(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + \dfrac{(-1)^{n+1} x^n}{n}$
$\dfrac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$
$\sqrt{1+x}$	$1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} - \dfrac{5x^4}{128} + \cdots + \dfrac{(-1)^{n+1}(2n-3)!! x^n}{2^n n!}$
$(1+x)^\alpha$	$1 + \alpha x + \dfrac{\alpha(\alpha-1)x^2}{2!} + \dfrac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \cdots + \dbinom{\alpha}{n} x^n$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

The Taylor polynomials of degree n for $f(x)$ at $x = a$.

$f(x)$	Taylor polynomial

$\cos x; \ a = \pi \qquad -1 + \dfrac{(x-\pi)^2}{2!} - \dfrac{(x-\pi)^4}{4!} + \cdots + \dfrac{(-1)^{k+1}(x-\pi)^{2k}}{(2k)!}$

$e^x; \ a = 2 \qquad e^2 + e^2(x-2) + \dfrac{e^2(x-2)^2}{2!} + \cdots + \dfrac{e^2(x-2)^n}{n!}$

$\dfrac{1}{x}; \ x = 1 \qquad 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n$

$\dfrac{1}{2+x}; \ a = 0 \qquad \dfrac{1}{2} - \dfrac{x}{4} + \dfrac{x^2}{8} - \dfrac{x^3}{16} + \cdots + \dfrac{(-1)^n x^n}{2^{n+1}}$

$\dfrac{1}{3-2x}; \ x = 1 \qquad 1 + 2(x-1) + 4(x-1)^2 + 8(x-1)^3 + \cdots + 2^n(x-1)^n$

$\sqrt{100-2x}; \ a = 0 \qquad 10 - \dfrac{x}{10} - \dfrac{x^2}{2000} - \dfrac{x^3}{200000} - \cdots - \dfrac{(2n-3)!!x^n}{10^{2n-1}n!}$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation

Definition (Taylor series)

Let $f(x)$ be an infinitely differentiable function. The **Taylor series** of $f(x)$ at $x = a$ is the infinite power series

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

The following table shows the Taylor series for $f(x)$ at $x = a$.

$f(x)$	Taylor series
$e^x;\ a = 0$	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$
$\cos x;\ a = 0$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$
$\sin x;\ a = \pi$	$-(x - \pi) + \dfrac{(x - \pi)^3}{3!} - \dfrac{(x - \pi)^5}{5!} + \cdots$
$\ln x;\ a = 1$	$(x - 1) - \dfrac{(x - 1)^2}{2} + \dfrac{(x - 1)^3}{3} - \dfrac{(x - 1)^4}{4} + \cdots$
$\sqrt{1 + x};\ a = 0$	$1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} - \dfrac{5x^4}{128} + \cdots$
$\dfrac{1}{\sqrt{1 + x}};\ a = 0$	$1 - \dfrac{x}{2} + \dfrac{3x^2}{8} - \dfrac{5x^3}{16} + \dfrac{35x^4}{128} - \dfrac{63x^5}{256} + \cdots$
$(1 + x)^{\alpha};\ a = 0$	$1 + \alpha x + \dfrac{\alpha(\alpha - 1)x^2}{2!} + \dfrac{\alpha(\alpha - 1)(\alpha - 2)x^3}{3!} + \cdots$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

$$e^x; \qquad \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x; \qquad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x; \qquad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\ln(1+x); \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{1}{1-x}; \qquad \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^\alpha; \qquad \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \cdots$$

$$\tan^{-1} x; \qquad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\sin^{-1} x; \qquad \sum_{k=0}^{\infty} \frac{(2k)! x^{2k+1}}{4^k (k!)^2 (2k+1)} = x + \left(\frac{1}{2}\right) \frac{x^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right) \frac{x^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) \frac{x^7}{7} + \cdots$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Suppose $T(x)$ is the Taylor series of $f(x)$ at $x = 0$. Then for any positive integer k, the Taylor series for $f(x^k)$ at $x = 0$ is $T(x^k)$.

Example

$$
\begin{array}{ll}
f(x) & \text{Taylor series at } x = 0 \\[1em]
\dfrac{1}{1 + x^2} & 1 - x^2 + x^4 - x^6 + \cdots \\[1em]
\dfrac{1}{\sqrt{1 - x^2}} & 1 + \dfrac{x^2}{2} + \dfrac{3x^4}{8} + \dfrac{5x^6}{16} + \dfrac{35x^8}{128} + \cdots \\[1em]
\dfrac{\sin x^2}{x^2} & 1 - \dfrac{x^4}{3!} + \dfrac{x^8}{5!} - \dfrac{x^{12}}{7!} + \cdots
\end{array}
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Suppose the Taylor series for $f(x)$ at $x = 0$ is

$$T(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Then the Taylor series for $f'(x)$ is

$$T'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

Find the Taylor series of the following functions.

1. $\dfrac{1}{(1+x)^2}$

2. $\tan^{-1} x$

Solution

1. Let $F(x) = -\dfrac{1}{1+x}$ so that $F'(x) = \dfrac{1}{(1+x)^2}$. The Taylor series for $F(x)$ at $x = 0$ is

$$T(x) = -1 + x - x^2 + x^3 - x^4 + \cdots.$$

Therefore the Taylor series for $F'(x) = \dfrac{1}{(1+x)^2}$ is

$$T'(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Solution

2. *Suppose the Taylor series for $f(x) = \tan^{-1} x$ at $x = 0$ is*

$$T(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots.$$

Now comparing $T'(x)$ with the Taylor series for $f'(x) = \dfrac{1}{1+x^2}$ which takes the form

$$1 - x^2 + x^4 - x^6 + \cdots,$$

we obtain the values of $a_1, a_2, a_3, \ldots$ and get

$$T(x) = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Since $a_0 = f(0) = 0$, we have

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Suppose the Taylor series for $f(x)$ and $g(x)$ at $x = 0$ are

$$S(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

$$T(x) = \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots,$$

respectively. Then the Taylor series for $f(x)g(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

Limits   Derivatives
Differentiation   Mean value theorem
Integration   **Application of Differentiation**

Proof.

The coefficient of x^n of the Taylor series of $f(x)g(x)$ at $x = 0$ is

$$
\begin{aligned}
\frac{(fg)^{(n)}(0)}{n!} &= \sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \quad \text{(Leibniz's formula)} \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot \frac{g^{(n-k)}(0)}{(n-k)!} \\
&= \sum_{k=0}^{n} a_k b_{n-k}
\end{aligned}
$$

□

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

1. The Taylor series for $e^{4x} \ln(1+x)$ is

$$
\left(1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)
$$

$$
= x + \left(-\frac{1}{2} + 4\right)x^2 + \left(\frac{1}{3} + 4 \cdot \left(-\frac{1}{2}\right) + 8\right)x^3 + \cdots
$$

$$
= x + \frac{7x^2}{2} + \frac{19x^3}{3} + \cdots
$$

2. The Taylor series for $\dfrac{\tan^{-1} x}{\sqrt{1 - x^2}}$ is

$$
\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)\left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \cdots\right)
$$

$$
= x + \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{3}{4} - \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{5}\right)x^5 + \cdots
$$

$$
= x + \frac{x^3}{6} + \frac{49x^5}{120} + \cdots
$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Theorem

Suppose $f(x)$ and $g(x)$ are infinitely differentiable functions and the Taylor series of $f(x)$ and $g(x)$ at $x = 0$ are

$$a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \cdots$$

and

$$b_k x^k + b_{k+1} x^{k+1} + b_{k+2} x^{k+2} + \cdots$$

where $b_k \neq 0$. Then

$$\begin{aligned}
\lim_{x \to 0} \frac{f(x)}{g(x)} &= \lim_{x \to 0} \frac{a_k + a_{k+1} x + a_{k+2} x^2 + \cdots}{b_k + b_{k+1} x + b_{k+2} x^2 + \cdots} \\
&= \frac{a_k}{b_k}
\end{aligned}$$

Limits
Differentiation
Integration
Derivatives
Mean value theorem
Application of Differentiation

Proof.

The assumptions on $f(x)$ and $g(x)$ imply that

$$f(0) = f'(0) = f''(0) = \cdots = f^{(k-1)}(0) = 0; \ f^{(k)}(0) = a_k$$
$$g(0) = g'(0) = g''(0) = \cdots = g^{(k-1)}(0) = 0; \ g^{(k)}(0) = b_k$$

Therefore, by L'Hopital's rule, we have

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f''(x)}{g''(x)} = \cdots = \lim_{x \to 0} \frac{f^{(k)}(x)}{g^{(k)}(x)} = \frac{a_k}{b_k}.$$

$\square$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example

1. $\quad \lim_{x \to 0} \dfrac{\ln(1+x) - x\sqrt{1-x}}{x - \sin x}$

$= \quad \lim_{x \to 0} \dfrac{(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots) - x(1 - \frac{x}{2} - \frac{x^2}{8} + \cdots)}{x - (x - \frac{x^3}{6} + \cdots)}$

$= \quad \lim_{x \to 0} \dfrac{\frac{11x^3}{24} + \cdots}{\frac{x^3}{6} + \cdots}$

$= \quad \dfrac{11}{4}$

2. $\quad \lim_{x \to 0} \left(\dfrac{e^x}{x} - \dfrac{1}{\tan x} \right) = \lim_{x \to 0} \dfrac{e^x \sin x - x \cos x}{x \sin x}$

$= \quad \lim_{x \to 0} \dfrac{(1 + x + \frac{x^2}{2} + \cdots)(x - \frac{x^3}{6} + \cdots) - x(1 - \frac{x^2}{2} + \cdots)}{x(x - \frac{x^3}{6} + \cdots)}$

$= \quad \lim_{x \to 0} \dfrac{(x + x^2 + \frac{x^3}{3} + \cdots) - (x - \frac{x^3}{2} + \cdots)}{x^2 - \frac{x^4}{6} + \cdots}$

$= \quad \lim_{x \to 0} \dfrac{x^2 + \frac{5x^3}{6} + \cdots}{x^2 - \frac{x^4}{6} + \cdots}$

$= \quad 1$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Curve sketching

To sketch the graph of $y = f(x)$, one first finds

- Domain: The values of x where $f(x)$ is defined.
- x-intercepts: The values of x such that $f(x) = 0$.
- y-intercept: $f(0)$
- Horizontal asymptotes:
  If $\lim\limits_{x \to -\infty/+\infty} f(x) = b$, then $y = b$ is a horizontal asymptote.
- Vertical asymptotes:
  If $\lim\limits_{x \to a^-/a^+} f(x) = -\infty/+\infty$, then $x = a$ is a vertical asymptote.

Limits   Derivatives
Differentiation   Mean value theorem
Integration   Application of Differentiation

Example 1: $f(x) = \dfrac{3x + 5}{x + 2}$
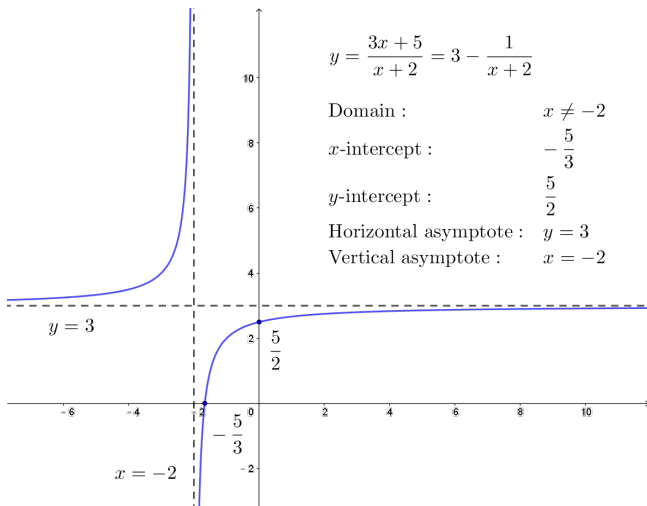
$$y = \frac{3x + 5}{x + 2} = 3 - \frac{1}{x + 2}$$

Domain : $x \neq -2$

x-intercept : $-\dfrac{5}{3}$

y-intercept : $\dfrac{5}{2}$

Horizontal asymptote : $y = 3$

Vertical asymptote : $x = -2$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 2: $f(x) = \dfrac{x^2 + 2}{x^2 + 1}$

$$y = \frac{x^2 + 2}{x^2 + 1} = 1 + \frac{1}{x^2 + 1}$$

Domain : $\qquad\qquad\qquad$ $\mathbb{R}$

x-intercept : $\qquad\qquad$ none

y-intercept : $\qquad\qquad$ 2

Horizontal asymptote : $\quad$ $y = 1$

Vertical asymptote : $\qquad$ none

Limits   Derivatives
Differentiation   Mean value theorem
Integration   Application of Differentiation

Example 3: $f(x) = \dfrac{x}{|x| + 1}$

$y = 1$

$y = \dfrac{x}{|x| + 1}$

$y = -1$

Domain :	$\mathbb{R}$
x-intercept :	0
y-intercept :	0
Horizontal asymptote :	$y = -1, y = 1$
Vertical asymptote :	none

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 4: $f(x) = |\ln|x||$

$$y = |\ln|x||$$

Domain :	$x \neq 0$
x-intercept :	± 1
y-intercept :	none
Horizontal asymptote :	none
Vertical asymptote :	$x = 0$

$$y = \ln|x|$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Definition (Oblique asymptote)

If

$$\lim_{x \to -\infty/+\infty} (f(x) - (ax + b)) = 0,$$

we say that $y = ax + b$ is an oblique asymptote of $y = f(x)$.

$y = f(x)$

$y = ax + b$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
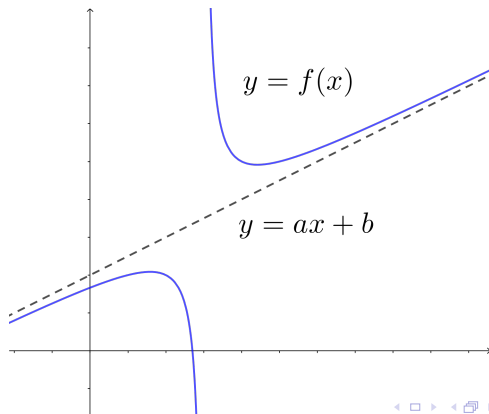
Example 5: $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$.

Note that $\dfrac{x^2 - 3x - 4}{x - 2} = \dfrac{x^2 - 2x - (x - 2) - 6}{x - 2} = x - 1 - \dfrac{6}{x - 2}$.

$$y = x - 1$$

$$y = \frac{x^2 - 3x - 4}{x - 2}$$

$$x = 2$$

Domain :	$x \neq 2$
x-intercept :	$-1, 4$
y-intercept :	2
Vertical asymptote :	$x = 2$
Oblique asymptote :	$y = x - 1$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
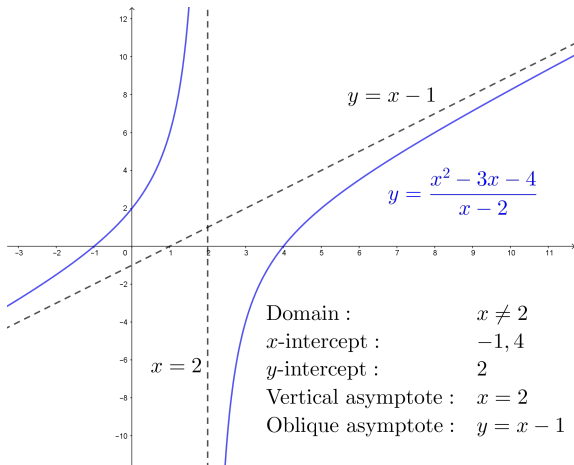
Definition

Let $f(x)$ be a continuous function. We say that $f(x)$ has a

1. **local maximum** at $x = a$ if there exists $\delta > 0$ such that $f(x) \leq f(a)$ for any $x \in (a - \delta, a + \delta)$.

2. **local minimum** at $x = a$ if there exists $\delta > 0$ such that $f(x) \geq f(a)$ for any $x \in (a - \delta, a + \delta)$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
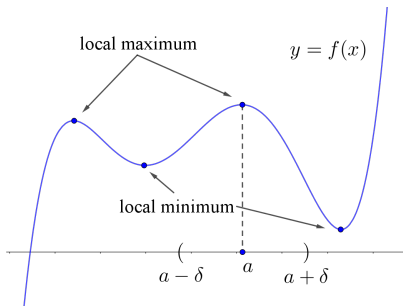
Theorem

Let $f(x)$ be a continuous function. Suppose $f(x)$ has local maximum or local minimum at $x = a$. Then either

1. $f'(a) = 0$, or
2. $f'(x)$ does not exist at $x = a$.

$f'(x) = 0$

$f'(x)$ does not exist

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
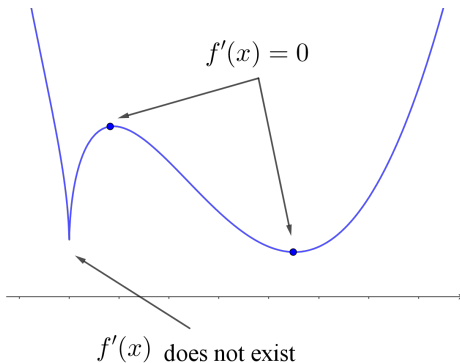
Theorem (First derivative test)

Let $f(x)$ be a continuous function and $f'(a) = 0$ or $f'(a)$ does not exist. Suppose there is $\delta > 0$ such that

1

	$a - \delta < x < a$	$a < x < a + \delta$
$f'(x)$	$+$	$-$

Then $f(x)$ has a local maximum at $x = a$.

2

	$a - \delta < x < a$	$a < x < a + \delta$
$f'(x)$	$-$	$+$

Then $f(x)$ has a local minimum at $x = a$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
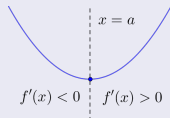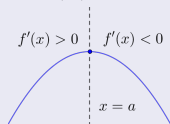
Theorem (Second derivative test)

Let $f(x)$ be a differentiable function and $f'(a) = 0$.

1. *If $f''(a) < 0$, then $f(x)$ has a local maximum at $x = a$.*

$$f''(a) < 0$$

2. *If $f''(a) > 0$, then $f(x)$ has a local minimum at $x = a$.*

$$f''(a) > 0$$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
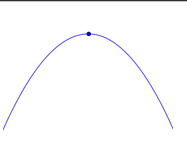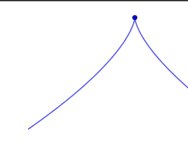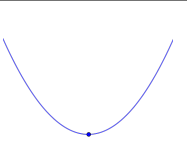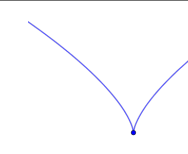
Definition (Turning point)

We say that $f(x)$ has a **turning point** at $x = a$ if $f'(x)$ changes
sign at $x = a$.

If $f(x)$ has a turning point at $x = a$, then either $f'(a) = 0$ or
$f'(x)$ does not exist.

Turning point	$f'(a) = 0$	$f'(a)$ does not exist
Relative maximum		
Relative minimum		

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 6: $f(x) = \dfrac{x-3}{x^2+4x-5}$

$f(x) = \dfrac{x-3}{(x-1)(x+5)}$, $x \neq -5, 1$

$f'(x) = \dfrac{(x^2+4x-5)(1)-(x-3)(2x+4)}{(x-1)^2(x+5)^2} = -\dfrac{(x+1)(x-7)}{(x-1)^2(x+5)^2}$

Thus $f'(x) = 0$ when $x = -1, 7$.

	$x < -5$	$-5 < x < -1$	$-1 < x < 1$	$1 < x < 7$	$x > 7$
$f'(x)$	$-$	$-$	$+$	$+$	$-$

$(-1, \frac{1}{2})$ is a minimum point and $(7, \frac{1}{18})$ is a maximum point.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example: $f(x) = \dfrac{x-3}{x^2+4x-5}$.

$$y = \dfrac{x-3}{x^2+4x-5}$$

Domain :	$x \neq -5, 1$
x-intercept :	3
y-intercept :	$\dfrac{3}{5}$
Horizontal asymptote :	$y = 0$
Vertical asymptote :	$x = -5, x = 1$
Turning points :	$(-1, \dfrac{1}{2}), (7, \dfrac{1}{18})$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
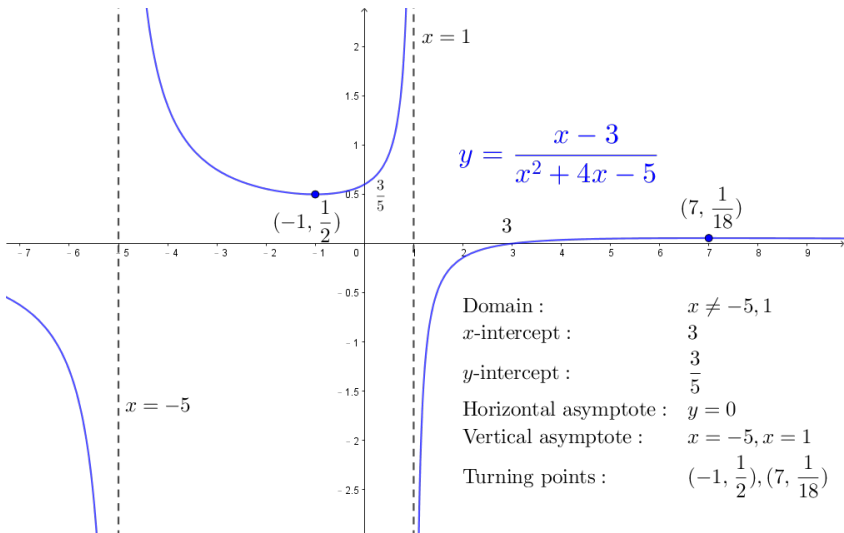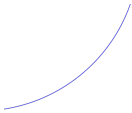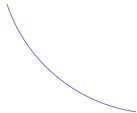
Definition (Concavity)

We say that $f(x)$ is

1. **Concave upward** on (a, b) if $f''(x) > 0$ on (a, b).

2. **Concave downward** on (a, b) if $f''(x) < 0$ on (a, b).

	$f'(x) > 0$	$f'(x) < 0$
Concave upward ($f''(x) > 0$)		
Concave downward ($f''(x) < 0$)		

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Definition (Inflection point)

We say that $f(x)$ has an **inflection point** at $x = a$ if $f''(x)$ changes sign at $x = a$.

If $f(x)$ has an inflection point at $x = a$, then ether $f''(a) = 0$ or $f''(a)$ does not exist.

$f''(x) > 0$ $f''(x) < 0$ $f''(x) < 0$ $f''(x) > 0$
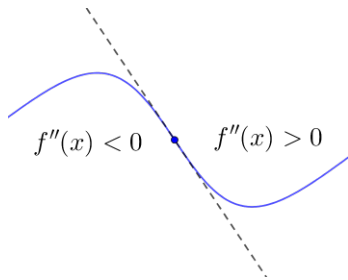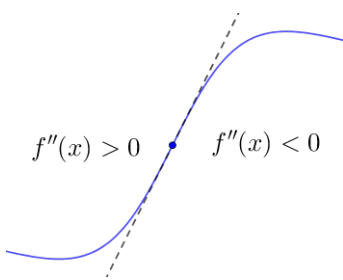
Example 7: $f(x) = |x+1|(3-x)$

$$f(x) = |x+1|(3-x) = \begin{cases} (x+1)(x-3) & \text{if } x < -1 \\ -(x+1)(x-3) & \text{if } x \geq -1 \end{cases}$$

$y = |x+1|(3-x)$

$(1, 4)$

Domain :	$\mathbb{R}$
x-intercept :	$-1, 3$
y-intercept :	3
Asymptotes :	none
Turning point :	$(-1, 0), (1, 4)$
Inflecction point :	$(-1, 0)$

$y = (x+1)(3-x)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
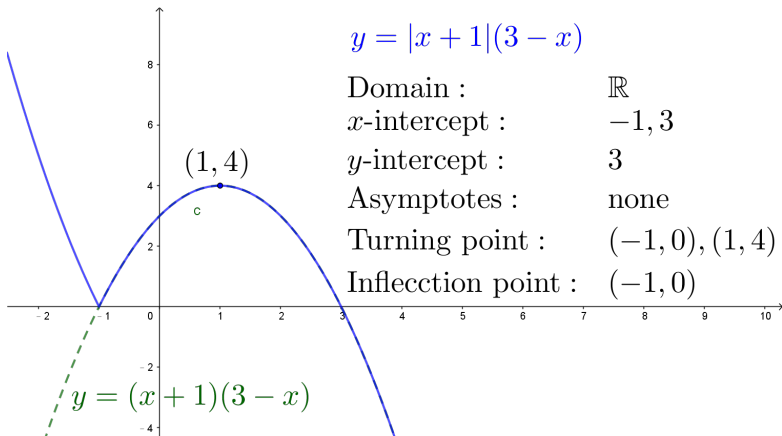
Example 8: $f(x) = x + \dfrac{1}{|x|}$

Since $\lim\limits_{x \to \pm\infty} (f(x) - x) = \lim\limits_{x \to \pm\infty} \dfrac{1}{|x|} = 0$,

$y = f(x)$ has an oblique asymptote $y = x$.

When $x < 0$, $f(x) = x - \dfrac{1}{x}$.

$f'(x) = 1 + \dfrac{1}{x^2}$

$f''(x) = -\dfrac{2}{x^3}$

When $x > 0$, $f(x) = x + \dfrac{1}{x}$.

$f'(x) = 1 - \dfrac{1}{x^2}$

$f''(x) = \dfrac{2}{x^3}$

	$x < 0$	$0 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$+$	$+$	$+$

$f(x)$ has a minimum point at $x = 1$.

$f(x)$ has no inflection point.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 8: $f(x) = x + \dfrac{1}{|x|}$

$$y = x + \frac{1}{|x|}$$

$$y = x$$

$(1, 2)$

Domain :	$x \neq 0$
x-intercept :	-1
y-intercept :	none
Asymptotes :	$x = 0, y = x$
Turning point :	$(1, 2)$
Inflection point :	none

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 9: $f(x) = \dfrac{|2x+1|}{x-3}$

$x = 3$

$y = \dfrac{2x+1}{x-3}$

$y = \dfrac{|2x+1|}{x-3}$

$y = 2$

$y = -2$

Domain :	$x \neq 3$
x-intercept :	$-\dfrac{1}{2}$
y-intercept :	$-\dfrac{1}{3}$
Asymptotes :	$x = 3, y = -2, y = 2$
Turning point :	$\left(-\dfrac{1}{2}, 0\right)$
Inflection point :	$\left(-\dfrac{1}{2}, 0\right)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 10: $f(x) = 2 - (x-8)^{\frac{1}{3}}$

$f'(x) = -\dfrac{1}{3(x-8)^{\frac{2}{3}}}$

$f''(x) = \dfrac{2}{9(x-8)^{\frac{5}{3}}}$

$f'(x)$, $f''(x)$ do not exist at $x = 8$.

	$x < 8$	$x > 8$
$f'(x)$	$-$	$-$
$f''(x)$	$-$	$+$

$f(x)$ has no turning point.

$f(x)$ has an inflection point at $x = 8$.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 10: $f(x) = 2 - (x-8)^{\frac{1}{3}}$

$y = 2 - (x-8)^{\frac{1}{3}}$

Domain :	$\mathbb{R}$
x-intercept :	16
y-intercept :	4
Asymptotes :	none
Turning point :	none
Inflection point	$(8,2)$

$(8,2)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 11: $f(x) = \left| 1 - \sqrt{|x|} \right|$

$$y = \left| 1 - \sqrt{|x|} \right|$$

$$y = \sqrt{|x|} - 1$$

Domain :	$\mathbb{R}$
x-intercept :	$-1, 0, 1$
y-intercept :	0
Asymptotes :	none
Turning point :	$(-1, 0), (0, 1), (1, 0)$
Inflection point :	$(-1, 0), (1, 0)$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 12: $f(x) = \dfrac{x^2 + x - 2}{x^2}$

Domain: $x \neq 0$

$f(x) = \dfrac{x^2 + x - 2}{x^2} = 1 + \dfrac{x - 2}{x^2}$

$f(x)$ has a horizontal asymptote $y = 1$.

$f'(x) = \dfrac{x^2 - 2x(x-2)}{x^4} = \dfrac{x - 2(x-2)}{x^3} = -\dfrac{x-4}{x^3}$

$f'(x) = 0$ when $x = 4$

$f''(x) = -\dfrac{x^3 - 3x^2(x-4)}{x^6} = -\dfrac{x - 3(x-4)}{x^6} = \dfrac{2(x-6)}{x^4}$

$f''(x) = 0$ when $x = 6$.

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$

$(4, \frac{9}{8})$ is maximum point.

$(6, \frac{10}{9})$ is an inflection point.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 12: $f(x) = \dfrac{x^2 + x - 2}{x^2}$

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$

$(4, \dfrac{9}{8})$ $(6, \dfrac{10}{9})$

$y = 1$

$y = \dfrac{x^2 + x - 2}{x^2}$

-2 1

Domain : $\qquad\qquad$ $x \neq 0$
x-intercept : $\qquad\quad$ $-2, 1$
y-intercept : $\qquad\quad$ none
Vertical asymptote : $\quad$ $x = 0$
Horizontal asymptote : $\;$ $y = 1$
Maximum point : $\qquad$ $(4, \dfrac{9}{8})$
Inflection point : $\qquad$ $(6, \dfrac{10}{9})$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 12: $f(x) = \dfrac{x^2 + x - 2}{x^2}$
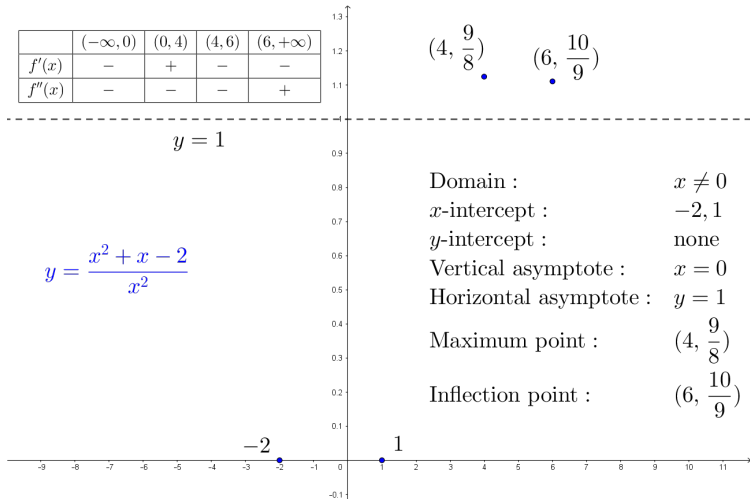
	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$

$\left(4, \dfrac{9}{8}\right)$ $\left(6, \dfrac{10}{9}\right)$

$y = 1$

$y = \dfrac{x^2 + x - 2}{x^2}$

Domain : $x \neq 0$
x-intercept : $-2, 1$
y-intercept :   none
Vertical asymptote : $x = 0$
Horizontal asymptote : $y = 1$
Maximum point : $\left(4, \dfrac{9}{8}\right)$
Inflection point : $\left(6, \dfrac{10}{9}\right)$

-2 1

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
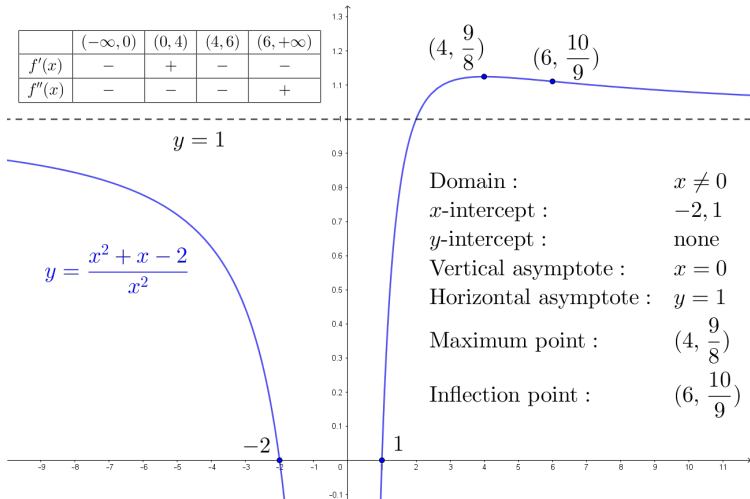
Example 13: $f(x) = \dfrac{x^3}{(x-2)^2}$

$f(x) = x + 4 + \dfrac{12x - 16}{(x-2)^2}$, $x \neq 2$

$f(x)$ has an oblique asymptote $y = x + 4$

$f'(x) = \dfrac{3x^2(x-2)^2 - 2(x-2)x^3}{(x-2)^4} = \dfrac{3x^2(x-2) - 2x^3}{(x-2)^3} = \dfrac{x^3 - 6x^2}{(x-2)^3}$

$f'(x) = 0$ when $x = 0, 6$

$f''(x) = \dfrac{(3x^2 - 12x)(x-2)^3 - 3(x-2)^2(x^3 - 6x^2)}{(x-2)^6} = \dfrac{24x}{(x-2)^4}$

$f''(x) = 0$ when $x = 0$.

	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$-$	$+$	$+$	$+$

$(6, \frac{27}{2})$ is minimum point.
$(0, 0)$ is an inflection point.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 13: $f(x) = \dfrac{x^3}{(x-2)^2}$

Domain :	$x \neq 2$	
x-intercept :	0	
y-intercept :	0	
Vertical asymptote :	$x = 2$	
Oblique asymptote :	$y = x + 4$	
Minimum point :	$(6, \dfrac{27}{2})$	
Inflection point :	$(0, 0)$	

$y = \dfrac{x^3}{(x-2)^2}$

$(6, \dfrac{27}{2})$

$y = x + 4$

$(0, 0)$

$x = 2$

	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$-$	$+$	$+$	$+$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 13: $f(x) = \dfrac{x^3}{(x-2)^2}$

Domain : $\quad x \neq 2$

x-intercept : $\quad 0$

y-intercept : $\quad 0$

Vertical asymptote : $\quad x = 2$

Oblique asymptote : $\quad y = x + 4$

Minimum point : $\quad (6, \dfrac{27}{2})$

Inflection point : $\quad (0, 0)$

	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$-$	$+$	$+$	$+$

$y = \dfrac{x^3}{(x-2)^2}$

$(6, \dfrac{27}{2})$

$y = x + 4$

$(0, 0)$

$x = 2$

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation
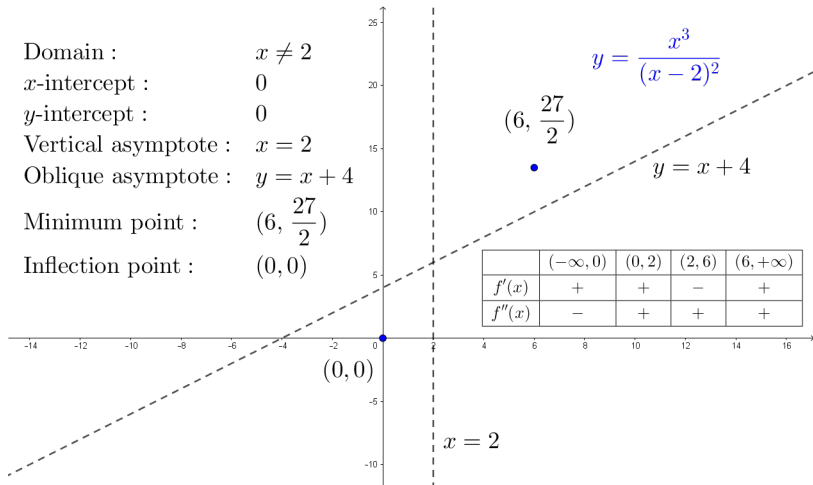
Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

First

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}}{x} = \lim_{x \to \pm\infty} \left(1 - \frac{3}{x}\right)^{\frac{2}{3}} = 1$$

and

$$
\begin{aligned}
\lim_{x \to \pm\infty} (f(x) - x) &= \lim_{x \to \pm\infty} x\left(\left(1 - \frac{3}{x}\right)^{\frac{2}{3}} - 1\right) \\
&= \lim_{h \to 0} \frac{(1 - 3h)^{\frac{2}{3}} - 1}{h} \\
&= -2
\end{aligned}
$$

Thus $y = x - 2$ is an oblique asymptote.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

$$
\begin{aligned}
f'(x) &= \frac{1}{3}x^{-\frac{2}{3}}(x-3)^{\frac{2}{3}} + \frac{2}{3}x^{\frac{1}{3}}(x-3)^{-\frac{1}{3}} \\
&= \frac{x-1}{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}}}
\end{aligned}
$$

$f'(x) = 0$ when $x = 1$ and $f'(x)$ does not exist when $x = 0, 3$.

$$
\begin{aligned}
f''(x) &= \frac{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}} - (\frac{2}{3}x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}})(x-1)}{x^{\frac{4}{3}}(x-3)^{\frac{2}{3}}} \\
&= \frac{3x(x-3) - (2(x-3)+x)(x-1)}{3x^{\frac{5}{3}}(x-3)^{\frac{4}{3}}} \\
&= -\frac{2}{x^{\frac{5}{3}}(x-3)^{\frac{4}{3}}}
\end{aligned}
$$

$f''(x)$ does not exist when $x = 0, 3$.

	$(-\infty, 0)$	$(0,1)$	$(1,3)$	$(3, +\infty)$
$f'(x)$	+	+	−	+
$f''(x)$	+	−	−	−

$(1, 2^{\frac{2}{3}})$ is a maximum point.

$(3, 0)$ is a minimum point.

$(0, 0)$ is an inflection point.

Limits
Differentiation
Integration

Derivatives
Mean value theorem
Application of Differentiation

Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

$y = x - 2$

$(1, 2^{\frac{2}{3}})$

	$(-\infty, 0)$	$(0,1)$	$(1,3)$	$(3,+\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$-$	$-$

$y = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

3

Domain : $\mathbb{R}$

x-intercept : $0, 3$

y-intercept : 0

Oblique asymptote : $y = x - 2$

Turning points : $(1, 2^{\frac{2}{3}}), (3, 0)$

Inflection point : $(0, 0)$

Limits
Derivatives
Differentiation
Mean value theorem
Integration
Application of Differentiation

Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$-$	$-$

$y = x - 2$

$(1, 2^{\frac{2}{3}})$

$y = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

Domain :	$\mathbb{R}$
x-intercept :	$0, 3$
y-intercept :	0
Oblique asymptote :	$y = x - 2$
Turning points :	$(1, 2^{\frac{2}{3}}), (3, 0)$
Inflection point :	$(0, 0)$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration
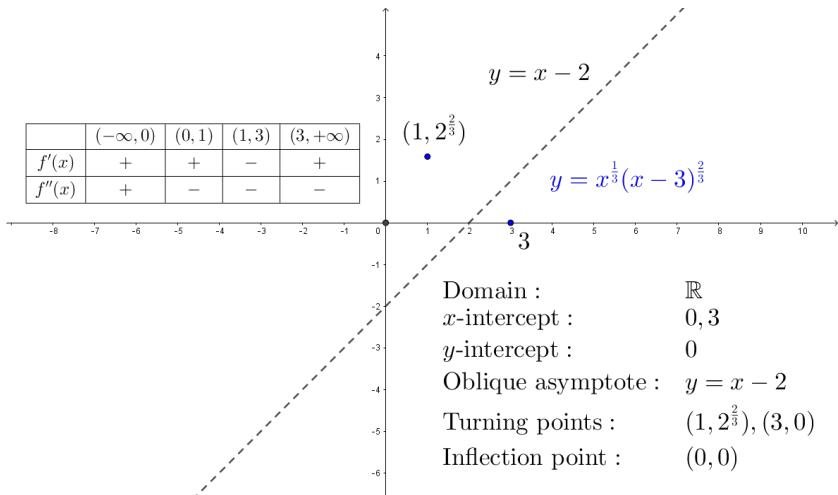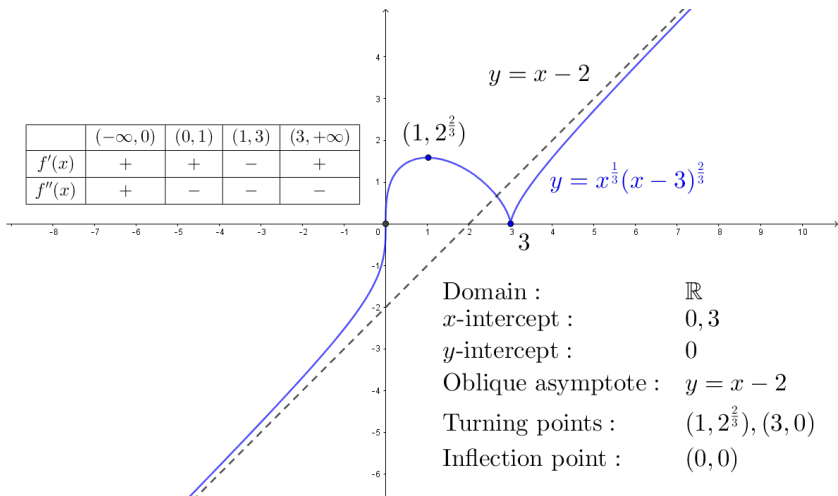
Indefinite integral and substitution

Definition

Let $f(x)$ be a continuous function. A **primitive function**, or an
anti-derivative, of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

The collection of all anti-derivatives of $f(x)$ is called the **indefinite
integral** of $f(x)$ and is denoted by

$$\int f(x)dx.$$

The function $f(x)$ is called the **integrand** of the integral.

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Note: Anti-derivative of a function is not unique. If $F(x)$ is an anti-derivative of f, then $F(x) + C$ is an anti-derivative of $f(x)$ for any constant C. Moreover, any anti-derivative of $f(x)$ is of the form $F(x) + C$ and we write

$$\int f(x)dx = F(x) + C$$

where C is arbitrary constant called the **integration constant**. Note that $\int f(x)dx$ is not a single function but a collection of functions.

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

Let $f(x)$ and $g(x)$ be continuous functions and k be a constant.

① $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

② $\int kf(x)dx = k \int f(x)dx$

Theorem (formulas for indefinite integrals)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$$

$$\int e^x dx = e^x + C; \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C; \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C; \qquad \int \csc x \cot x dx = -\csc x + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

1. $\displaystyle\int (x^3 - x + 5)dx \qquad = \quad \dfrac{x^4}{4} - \dfrac{x^2}{2} + 5x + C$

2. $\displaystyle\int \dfrac{(x+1)^2}{x}dx \qquad = \quad \displaystyle\int \dfrac{x^2 + 2x + 1}{x}dx$

$\qquad\qquad\qquad\qquad = \quad \displaystyle\int \left(x + 2 + \dfrac{1}{x}\right)dx$

$\qquad\qquad\qquad\qquad = \quad \dfrac{x^2}{2} + 2x + \ln|x| + C$

3. $\displaystyle\int \dfrac{3x^2 + \sqrt{x} - 1}{\sqrt{x}}dx \quad = \quad \displaystyle\int \left(3x^{3/2} + 1 - x^{-1/2}\right)dx$

$\qquad\qquad\qquad\qquad = \quad \dfrac{6}{5}x^{\frac{5}{2}} + x - 2x^{\frac{1}{2}} + C$

4. $\displaystyle\int \left(\dfrac{3\sin x}{\cos^2 x} - 2e^x\right)dx \quad = \quad \displaystyle\int (3\sec x \tan x - 2e^x)\,dx$

$\qquad\qquad\qquad\qquad = \quad 3\sec x - 2e^x + C$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Suppose we want to compute

$$\int x\sqrt{x^2+4}\,dx$$

First we let

$$u = x^2 + 4.$$

We may formally write

$$du = \frac{du}{dx}\,dx = \left[\frac{d}{dx}(x^2+4)\right]dx = 2xdx$$

Here du is called the differential of u defined as $\dfrac{du}{dx}\,dx$. Thus the integral is

$$\begin{aligned}
\int x\sqrt{x^2+4}\,dx &= \frac{1}{2}\int\sqrt{x^2+4}(2xdx) = \frac{1}{2}\int\sqrt{u}\,du \\
&= \frac{u^{\frac{3}{2}}}{3} + C = \frac{(x^2+4)^{\frac{3}{2}}}{3} + C
\end{aligned}$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

$$
\begin{aligned}
\int x\sqrt{x^2+4}\,dx &= \int \sqrt{x^2+4}\,d\left(\frac{x^2}{2}\right) \\
&= \frac{1}{2}\int \sqrt{x^2+4}\,dx^2 \\
&= \frac{1}{2}\int \sqrt{x^2+4}\,d(x^2+4) \\
&= \frac{(x^2+4)^{\frac{3}{2}}}{3} + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

Let $f(x)$ be a continuous function defined on $[a, b]$. Suppose there exists a differentiable function $u = \varphi(x)$ and continuous function $g(u)$ such that $f(x) = g(\varphi(x))\varphi'(x)$ for any $x \in (a, b)$. Then

$$
\begin{aligned}
\int f(x)dx &= \int g(\varphi(x))\varphi'(x)dx \\
&= \int g(u)du
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

$$\int x^2 e^{x^3+1} dx$$

Let $u = x^3 + 1$,

then $du = 3x^2 dx$

$$= \frac{1}{3} \int e^u du$$

$$= \frac{e^u}{3} + C$$

$$= \frac{e^{x^3+1}}{3} + C$$

$$\int x^2 e^{x^3+1} dx$$

$$= \int e^{x^3+1} d\left(\frac{x^3}{3}\right)$$

$$= \frac{1}{3} \int e^{x^3+1} dx^3$$

$$= \frac{1}{3} \int e^{x^3+1} d(x^3+1)$$

$$= \frac{e^{x^3+1}}{3} + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

$$\int \cos^4 x \sin x dx$$

Let $u = \cos x$,

then $du = -\sin x dx$

$$= -\int u^4 du$$

$$= -\frac{u^5}{5} + C$$

$$= -\frac{\cos^5 x}{5} + C$$

$$\int \cos^4 x \sin x dx$$

$$= \int \cos^4 x d(-\cos x)$$

$$= -\int \cos^4 x d\cos x$$

$$= -\frac{\cos^5 x}{5} + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

$$\int \frac{dx}{x \ln x}$$

Let $u = \ln x$,

then $du = \dfrac{dx}{x}$

$$= \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \ln |\ln x| + C$$

$$\int \frac{dx}{x \ln x}$$

$$= \int \frac{d \ln x}{\ln x}$$

$$= \ln |\ln x| + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

$$\int \frac{dx}{e^x + 1}$$

Let $u = 1 + e^{-x}$,

then $du = -e^{-x}dx$

$$= \int \frac{e^{-x}dx}{1 + e^{-x}}$$

$$= -\int \frac{du}{u}$$

$$= -\ln u + C$$

$$= -\ln(1 + e^{-x}) + C$$

$$= x - \ln(1 + e^x) + C$$

$$\int \frac{dx}{e^x + 1}$$

$$= \int \left(1 - \frac{e^x}{1 + e^x}\right) dx$$

$$= x - \int \frac{de^x}{1 + e^x}$$

$$= x - \ln(1 + e^x) + C$$

Example

$$\int \frac{dx}{1+\sqrt{x}}$$

Let $u = 1 + \sqrt{x}$,

then $du = \dfrac{dx}{2\sqrt{x}}$

$$= 2\int \frac{(u-1)du}{u}$$

$$= 2\int \left(1 - \frac{1}{u}\right) du$$

$$= 2u - 2\ln u + C'$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

$$\int \frac{dx}{1+\sqrt{x}}$$

$$= \int \frac{\sqrt{x}\,dx}{\sqrt{x}(1+\sqrt{x})}$$

$$= 2\int \frac{\sqrt{x}\,d\sqrt{x}}{1+\sqrt{x}}$$

$$= 2\int \left(1 - \frac{1}{1+\sqrt{x}}\right) d\sqrt{x}$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Definite integral

Definition

Let $f(x)$ be a function on $[a, b]$. A **Partition** of $[a, b]$ is a set of finite points

$$P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$$

and we define

$$
\begin{aligned}
\Delta x_k &= x_k - x_{k-1}, \text{ for } k = 1, 2, \ldots, n \\
\|P\| &= \max_{1 \leq k \leq n} \{\Delta x_k\}
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Definition

Let $f(x)$ be a function on $[a, b]$. The **lower** and **upper Riemann sums** with respect to partition P are

$$\mathcal{L}(f, P) = \sum_{k=1}^{n} m_k \Delta x_k, \text{ and } \mathcal{U}(f, P) = \sum_{k=1}^{n} M_k \Delta x_k$$

where

$$m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}, \text{ and } M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Figure: Upper and lower Riemann sum

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Figure: Upper and lower Riemann sum

Limits
Differentiation
Integration
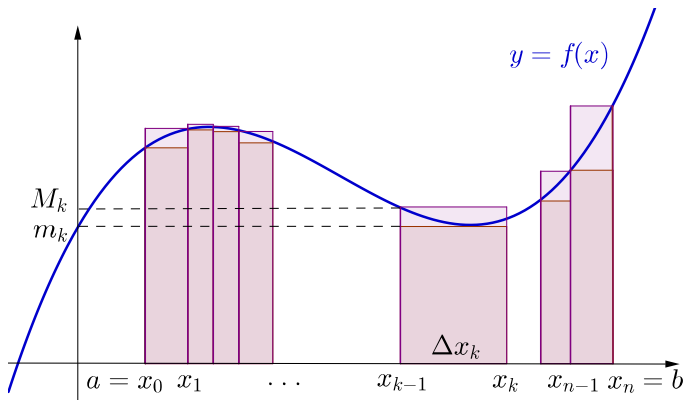
Integration
Techniques of Integration
More Techniques of Integration
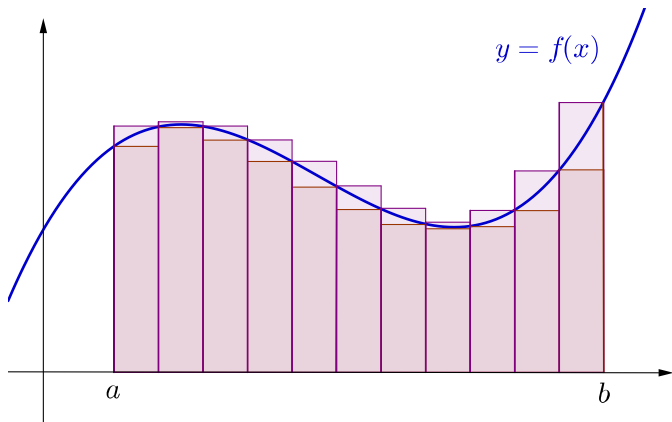
Definition (Riemann integral)

Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be a
real valued function defined on $[a, b]$. We say that $f(x)$ is
Riemann integrable on $[a, b]$ if the limits of $\mathcal{L}(f, P)$ and $\mathcal{U}(f, P)$
exist as $\|P\|$ tends to 0 and are *equal*. In this case, we define the
Riemann integral of $f(x)$ over $[a, b]$ by

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \mathcal{L}(f, P) = \lim_{\|P\| \to 0} \mathcal{U}(f, P).$$

Note: We say that $\lim_{\|P\| \to 0} \mathcal{L}(f, P) = L$ if for any $\varepsilon > 0$, there exists
$\delta = \delta(\varepsilon) > 0$ such that if $\|P\| < \delta$, then $|\mathcal{L}(f, P) - L| < \varepsilon$.

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$, $a < c < b$ and k be constants.

1. $\displaystyle\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

2. $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$

3. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

Suppose $f(x)$ is a continuous function on $[a, b]$. Then $f(x)$ is Riemann integrable on $[a, b]$ and we have

$$
\begin{aligned}
\int_a^b f(x)dx &= \lim_{n \to \infty} \sum_{k=1}^n f(x_k)\Delta x_k \\
&= \lim_{n \to \infty} \sum_{k=1}^n f\left(a + \frac{k}{n}(b-a)\right)\left(\frac{b-a}{n}\right).
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration
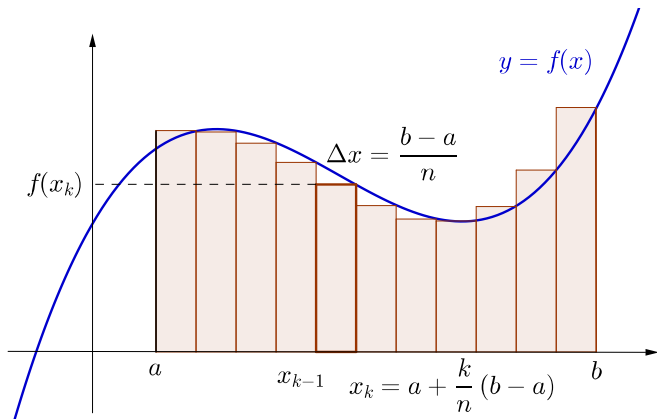
Figure: Formula for Riemann integral

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Use the formula for definite integral of continuous function to evaluate

$$\int_0^1 x^2 dx$$

Solution

$$
\begin{aligned}
\int_0^1 x^2 dx &= \lim_{n \to \infty} \sum_{k=1}^{n} \left(0 + \frac{k}{n}(1-0) \right)^2 \left(\frac{1-0}{n} \right) \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&= \frac{1}{3}
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Fundamental theorem of calculus

Theorem (Fundamental theorem of calculus)

First part: *Let $f(x)$ be a function which is continuous on $[a, b]$.*
Let $F : [a, b] \to \mathbb{R}$ be the function defined by

$$F(x) = \int_a^x f(t)dt$$

Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x).$$

for any $x \in (a, b)$. Put in another way, we have

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) \quad \text{for } x \in (a, b).$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem (Fundamental theorem of calculus)

Second part: Let $f(x)$ be a function which is continuous on $[a, b]$. Let $F(x)$ be a primitive function of $f(x)$, in other words, $F(x)$ is a continuous function on $[a, b]$ and $F'(x) = f(x)$ for any $x \in (a, b)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Let $f(x) = \sqrt{1-x^2}$. The graph of $y = f(x)$ is a unit semicircle centered at the origin. Using the formula for area of circular sectors, we calculate

$$F(x) = \int_0^x f(t)dt = \int_0^x \sqrt{1-t^2}dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2}.$$

By fundamental theorem of calculus, we know that $F(x)$ is an anti-derivative of $f(x)$. One may check this by differentiating $F(x)$ and get

$$
\begin{aligned}
F'(x) &= \frac{1}{2}\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) \\
&= \frac{1}{2}\left(\frac{1-x^2-x^2+1}{\sqrt{1-x^2}}\right) \\
&= \sqrt{1-x^2} \\
&= f(x)
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration
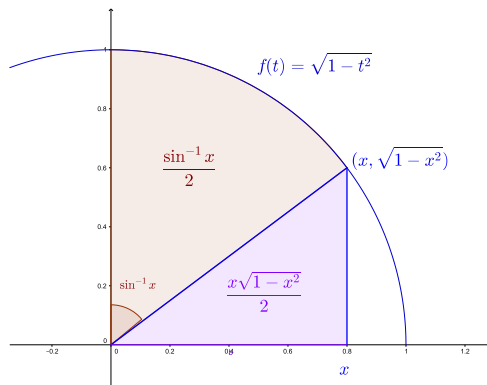
Figure: $\int_0^x \sqrt{1-t^2}dt = \dfrac{x\sqrt{1-x^2}}{2} + \dfrac{\sin^{-1} x}{2}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

1. $\displaystyle \int_1^3 (x^3 - 4x + 5)dx$ $= \left[\dfrac{x^4}{4} - 2x^2 + 5x \right]_1^3$

$= \left[\left(\dfrac{3^4}{4} - 2(3^2) + 5(3) \right) - \left(\dfrac{1^4}{4} - 2(1^2) + 5(1) \right) \right]$

$= 14$

2. $\displaystyle \int_{-3}^0 e^{2x+6}dx$ $= \left[\dfrac{e^{2x+6}}{2} \right]_{-3}^0$

$= \dfrac{e^6 - 1}{2}$

3. $\displaystyle \int_0^{\frac{\pi}{12}} \sec^2 3x \, dx$ $= \left[\dfrac{\tan 3x}{3} \right]_0^{\frac{\pi}{12}}$

$= \dfrac{\tan 3(\frac{\pi}{12}) - \tan 0}{3}$

$= \dfrac{1}{3}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

The fundamental theorem of calculus can be used to evaluate limit of series of a certain form.

Theorem

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{n}{n}\right) \right)$$

$$= \int_0^1 f(x)dx$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Find

1. $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \dfrac{1}{n+k} = \lim\limits_{n \to \infty} \left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n} \right)$

2. $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \dfrac{n}{n^2+k^2} = \lim\limits_{n \to \infty} \left(\dfrac{n}{n^2+1^2} + \dfrac{n}{n^2+2^2} + \cdots + \dfrac{n}{n^2+n^2} \right)$

3. $\lim\limits_{n \to \infty} \dfrac{1}{\sqrt{n}} \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{n+k}} = \lim\limits_{n \to \infty} \dfrac{1}{\sqrt{n}} \left(\dfrac{1}{\sqrt{n+1}} + \dfrac{1}{\sqrt{n+2}} + \cdots + \dfrac{1}{\sqrt{2n}} \right)$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

1. $\displaystyle \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n+k}$
$\begin{aligned}[t]
&= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}} \\
&= \int_0^1 \frac{1}{1+x}\, dx = [\ln(1+x)]_0^1 \\
&= \ln 2
\end{aligned}$

2. $\displaystyle \lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$
$\begin{aligned}[t]
&= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (\frac{k}{n})^2} \\
&= \int_0^1 \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_0^1 \\
&= \frac{\pi}{4}
\end{aligned}$

3. $\displaystyle \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{n+k}}$
$\begin{aligned}[t]
&= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1 + \frac{k}{n}}} \\
&= \int_0^1 \frac{1}{\sqrt{1+x}}\, dx = [2\sqrt{1+x}]_0^1 \\
&= 2(\sqrt{2} - 1)
\end{aligned}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Find $\displaystyle\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}$.

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

$$\ln\left(\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\ln\left(\frac{(n+1)(n+2)\cdots(2n)}{n^n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)\right)$$

$$= \int_0^1 \ln(1+x)dx$$

$$= \left[(1+x)\ln(1+x)-x\right]_0^1$$

$$= 2\ln 2 - 1$$

Therefore

$$\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n} = e^{2\ln 2 - 1} = \frac{4}{e} \approx 1.4715.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example (Definite integral and substitution)

1. $\displaystyle\int_3^5 x\sqrt{x^2-9}\,dx$

   Let $u = x^2 - 9$,

   When $x = 3$, $u = 0$

   When $x = 5$, $u = 16$

   $du = 2x\,dx$

   $= \dfrac{1}{2}\displaystyle\int_0^{16} \sqrt{u}\,du$

   $= \left[\dfrac{u^{\frac{3}{2}}}{3}\right]_0^{16}$

   $= \dfrac{64}{3}$

   $\displaystyle\int_3^5 x\sqrt{x^2-9}\,dx$

   $= \dfrac{1}{2}\displaystyle\int_3^5 \sqrt{x^2-9}\,d(x^2-9)$

   $= \dfrac{1}{3}\left[(x^2-9)^{\frac{3}{2}}\right]_3^5$

   $= \dfrac{64}{3}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example (Definite integral and substitution)

2.
$$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$,

When $x = 0$, $u = 0$

When $x = \pi^2$, $u = \pi$

$$du = \frac{dx}{2\sqrt{x}}$$

$$= 2\int_0^{\pi} \sin u\,du$$

$$= 2\left[-\cos u\right]_0^{\pi}$$

$$= 4$$

$$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

$$= 2\int_0^{\pi^2} \sin\sqrt{x}\,d\sqrt{x}$$

$$= 2\left[-\cos\sqrt{x}\right]_0^{\pi^2}$$

$$= 2\left[-\cos\sqrt{\pi^2} - (-\cos 0)\right]$$

$$= 4$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

We have the following formulas for derivatives of functions defined by integrals.

1. $\dfrac{d}{dx} \displaystyle\int_a^x f(t)dt = f(x)$

2. $\dfrac{d}{dx} \displaystyle\int_x^b f(t)dt = -f(x)$

3. $\dfrac{d}{dx} \displaystyle\int_a^{v(x)} f(t)dt = f(v)\dfrac{dv}{dx}$

4. $\dfrac{d}{dx} \displaystyle\int_{u(x)}^{v(x)} f(t)dt = f(v)\dfrac{dv}{dx} - f(u)\dfrac{du}{dx}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof.

1. This is the first part of fundamental theorem of calculus.

2. $\dfrac{d}{dx} \displaystyle\int_x^b f(t)dt \quad = \quad \dfrac{d}{dx}\left(-\displaystyle\int_b^x f(t)dt\right)$

   $\qquad\qquad\qquad = \quad -f(x)$

3. $\dfrac{d}{dx} \displaystyle\int_a^{v(x)} f(t)dt \quad = \quad \left(\dfrac{d}{dv}\displaystyle\int_a^{v(x)} f(t)dt\right)\dfrac{dv}{dx}$

   $\qquad\qquad\qquad = \quad f(v)\dfrac{dv}{dx}$

4. $\dfrac{d}{dx} \displaystyle\int_{u(x)}^{v(x)} f(t)dt \quad = \quad \dfrac{d}{dx}\left(\displaystyle\int_c^{v(x)} f(t)dt + \displaystyle\int_{u(x)}^c f(t)dt\right)$

   $\qquad\qquad\qquad = \quad \dfrac{d}{dx}\left(\displaystyle\int_c^{v(x)} f(t)dt - \displaystyle\int_c^{u(x)} f(t)dt\right)$

   $\qquad\qquad\qquad = \quad f(v)\dfrac{dv}{dx} - f(u)\dfrac{du}{dx}$

$\square$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Find $F'(x)$ for the the functions.

1. $F(x) = \int_1^x \sqrt{t} e^t dt$

2. $F(x) = \int_x^\pi \frac{\sin t}{t} dt$

3. $F(x) = \int_0^{\sin x} \sqrt{1 + t^4} dt$

4. $F(x) = \int_{-x}^{x^2} e^{t^2} dt$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. $\dfrac{d}{dx} \displaystyle\int_1^x \sqrt{t}e^t dt \qquad = \quad \sqrt{x}e^x$

2. $\dfrac{d}{dx} \displaystyle\int_x^\pi \dfrac{\sin t}{t} dt \qquad = \quad -\dfrac{\sin x}{x}$

3. $\dfrac{d}{dx} \displaystyle\int_0^{\sin x} \sqrt{1+t^4} dt \quad = \quad \sqrt{1+\sin^4 x}\,\dfrac{d}{dx}\sin x$

   $\qquad\qquad\qquad\qquad = \quad \cos x\sqrt{1+\sin^4 x}$

4. $\dfrac{d}{dx} \displaystyle\int_{-x}^{x^2} e^{t^2} dt \qquad = \quad e^{(x^2)^2}\dfrac{d}{dx}x^2 - e^{(-x)^2}\dfrac{d}{dx}(-x)$

   $\qquad\qquad\qquad\qquad = \quad 2xe^{x^4} + e^{x^2}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Trigonometric integrals

Techniques

Useful identities for trigonometric integrals.

1
- $\cos^2 x + \sin^2 x = 1$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$

2
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- $\cos x \sin x = \dfrac{\sin 2x}{2}$

3
- $\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))$
- $\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$
- $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Techniques

To evaluate

$$\int \cos^m x \sin^n x \, dx$$

where m, n are non-negative integers,

- *Case 1. If m is odd, use $\cos x \, dx = d \sin x$. (Substitute $u = \sin x$.)*
- *Case 2. If n is odd, use $\sin x \, dx = -d \cos x$. (Substitute $u = \cos x$.)*
- *Case 3. If both m, n are even, then use double angle formulas to reduce the power.*

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos x \sin x = \frac{\sin 2x}{2}$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Techniques

1. $\displaystyle\int \tan x\, dx = \ln|\sec x| + C$

2. $\displaystyle\int \cot x\, dx = \ln|\sin x| + C$

3. $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$

4. $\displaystyle\int \csc x\, dx = \ln|\csc x - \cot x| + C$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

We prove (1), (3) and the rest are left as exercise.

$$
\begin{aligned}
1. \int \tan x \, dx &= \int \frac{\sin x \, dx}{\cos x} \\
&= -\int \frac{d \cos x}{\cos x} \\
&= -\ln |\cos x| + C \\
&= \ln |\sec x| + C \\
3. \int \sec x \, dx &= \int \frac{\sec x (\sec x + \tan x) \, dx}{(\sec x + \tan x)} \\
&= \int \frac{(\sec^2 x + \sec x \tan x) \, dx}{(\sec x + \tan x)} \\
&= \int \frac{d(\tan x + \sec x)}{(\sec x + \tan x)} \\
&= \ln |\sec x + \tan x| + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Techniques

To evaluate

$$\int \sec^m x \tan^n x\, dx$$

where m, n are non-negative integers,

- *Case 1. If m is even, use $\sec^2 x\, dx = d\tan x$. (Substitute $u = \tan x$.)*
- *Case 2. If n is odd, use $\sec x \tan x\, dx = d\sec x$. (Substitute $u = \sec x$.)*
- *Case 3. If both m is odd and n is even, use $\tan^2 x = \sec^2 x - 1$ to write everything in terms of $\sec x$.*

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

1. $\int \sin^2 x \, dx$

2. $\int \cos^4 3x \, dx$

3. $\int \cos 2x \cos x \, dx$

4. $\int \cos 3x \sin 5x \, dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. $\displaystyle\int \sin^2 x \, dx \quad = \quad \int \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

2. $\displaystyle\int \cos^4 x \, dx \quad = \quad \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx$

$$= \quad \int \left(\frac{1 + 2\cos 2x + \cos^2 2x}{4} \right) dx$$

$$= \quad \frac{x}{4} + \frac{\sin 2x}{4} + \int \left(\frac{1 + \cos 4x}{8} \right) dx$$

$$= \quad \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

3. $\displaystyle\int \cos 2x \cos x \, dx \quad = \quad \frac{1}{2} \int \left(\cos 3x + \cos x \right) dx = \frac{\sin 3x}{6} + \frac{\sin x}{2} + C$

4. $\displaystyle\int \cos 3x \sin 5x \, dx \quad = \quad \frac{1}{2} \int \left(\sin 8x + \sin 2x \right) dx = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

1. $\displaystyle\int \cos x \sin^4 x\, dx$

2. $\displaystyle\int \cos^2 x \sin^3 x\, dx$

3. $\displaystyle\int \cos^4 x \sin^2 x\, dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. $\displaystyle\int \cos x \sin^4 x\,dx \;=\; \int \sin^4 x\,d\sin x = \frac{\sin^5 x}{5} + C$

2. $\displaystyle\int \cos^2 x \sin^3 x\,dx \;=\; -\int \cos^2 x (1 - \cos^2 x)\,d\cos x$

$$=\; -\int (\cos^2 x - \cos^4 x)\,d\cos x$$

$$=\; -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} C$$

3. $\displaystyle\int \cos^4 x \sin^2 x\,dx \;=\; \int \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{\sin 2x}{2}\right)^2 dx$

$$=\; \frac{1}{8}\int \left(\sin^2 2x + \cos 2x \sin^2 2x\right) dx$$

$$=\; \frac{1}{8}\int \left(\frac{1 - \cos 4x}{2}\right) dx + \frac{1}{16}\int \sin^2 2x\,d\sin 2x$$

$$=\; \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

1. $\displaystyle\int \sec^2 x \tan^2 x \, dx$

2. $\displaystyle\int \sec x \tan^3 x \, dx$

3. $\displaystyle\int \tan^3 x \, dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. $\displaystyle\int \sec^2 x \tan^2 x\, dx \quad = \quad \int \tan^2 x\, d\tan x = \frac{\tan^3 x}{3} + C$

2. $\displaystyle\int \sec x \tan^3 x\, dx \quad = \quad \int \tan^2 x\, d\sec x = \int (\sec^2 x - 1)d\sec x$

$$= \quad \frac{\sec^3 x}{3} - \sec x + C$$

3. $\displaystyle\int \tan^3 x\, dx \quad = \quad \int \tan x(\sec^2 x - 1)dx$

$$= \quad \int \tan x \sec^2 x\, dx - \int \tan x\, dx$$

$$= \quad \int \tan x\, d\tan x - \ln|\sec x|$$

$$= \quad \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Integration by parts

Techniques

Suppose the integrand is of the form $u(x)v'(x)$. Then we may evaluate the integration using the formula

$$\int uv' dx = uv - \int u'v dx.$$

The above formula is called integration by parts. It is usually written in the form

$$\int u\,dv = uv - \int v\,du.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

1. $\displaystyle\int xe^{3x}\,dx$

2. $\displaystyle\int x^2 \cos x\,dx$

3. $\displaystyle\int x^3 \ln x\,dx$

4. $\displaystyle\int \ln x\,dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. $\displaystyle \int xe^{3x}dx$ $\displaystyle = \frac{1}{3}\int xde^{3x} = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx$

$\displaystyle = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$

2. $\displaystyle \int x^2\cos xdx$ $\displaystyle = \int x^2 d\sin x$

$\displaystyle = x^2\sin x - \int \sin xdx^2$

$\displaystyle = x^2\sin x - 2\int x\sin xdx$

$\displaystyle = x^2\sin x + 2\int xd\cos x$

$\displaystyle = x^2\sin x + 2x\cos x - 2\int \cos xdx$

$\displaystyle = x^2\sin x + 2x\cos x - 2\sin x + C$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

$$3. \int x^3 \ln x \, dx \quad = \quad \frac{1}{4} \int \ln x \, dx^4$$

$$= \quad \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^4 \, d\ln x$$

$$= \quad \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^4 \left(\frac{1}{x}\right) dx$$

$$= \quad \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 \, dx$$

$$= \quad \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

$$4. \int \ln x \, dx \quad = \quad x \ln x - \int x \, d\ln x$$

$$= \quad x \ln x - \int dx$$

$$= \quad x \ln x - x + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

5. $\displaystyle\int_0^\pi x \sin x \, dx$

6. $\displaystyle\int_0^1 e^{\sqrt{x}} dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

5. $\displaystyle\int_0^\pi x\sin x\,dx$ $=$ $\displaystyle -\int_0^\pi x\,d\cos x$

$=$ $\displaystyle -[x\cos x]_0^\pi + \int_0^\pi \cos x\,dx$

$=$ $-(\pi\cos\pi - 0) + [\sin x]_0^\pi$

$=$ π

6. $\displaystyle\int_0^1 e^{\sqrt{x}}dx$ $=$ $\displaystyle 2\int_0^1 \sqrt{x}e^{\sqrt{x}}d\sqrt{x}$

$=$ $\displaystyle 2\int_0^1 \sqrt{x}de^{\sqrt{x}}$

$=$ $\displaystyle 2[\sqrt{x}e^{\sqrt{x}}]_0^1 - 2\int_0^1 e^{\sqrt{x}}d\sqrt{x}$

$=$ $2e - 2[e^{\sqrt{x}}]_0^1$

$=$ $2e - 2(e-1)$

$=$ 2

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

7. $\int \sin^{-1} x\, dx$

8. $\int \ln(1 + x^2)\, dx$

9. $\int \sec^3 x\, dx$

10. $\int e^x \sin x\, dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

$$7. \int \sin^{-1} x \, dx \quad = \quad x \sin^{-1} x - \int x \, d\sin^{-1} x$$

$$= \quad x \sin^{-1} x - \int \frac{x \, dx}{\sqrt{1 - x^2}}$$

$$= \quad x \sin^{-1} x + \frac{1}{2} \int \frac{d(1 - x^2)}{\sqrt{1 - x^2}}$$

$$= \quad x \sin^{-1} x + \sqrt{1 - x^2} + C$$

$$8. \int \ln(1 + x^2) \, dx \quad = \quad x \ln(1 + x^2) - \int x \, d\ln(1 + x^2)$$

$$= \quad x \ln(1 + x^2) - 2 \int \frac{x^2 \, dx}{1 + x^2}$$

$$= \quad x \ln(1 + x^2) - 2 \int \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= \quad x \ln(1 + x^2) - 2x + 2 \tan^{-1} x + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

$$9. \quad \int \sec^3 x\, dx = \int \sec x\, d\tan x$$

$$= \sec x \tan x - \int \tan x\, d\sec x$$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$\int \sec^3 x\, dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

$$10. \quad \int e^x \sin x \, dx = \int \sin x \, de^x$$

$$= e^x \sin x - \int e^x \, d\sin x$$

$$= e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x - \int \cos x \, de^x$$

$$= e^x \sin x - e^x \cos x + \int e^x \, d\cos x$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C'$$

$$\int e^x \sin x \, dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Reduction formula

Techniques

For integral of the forms

$$
\begin{aligned}
I_n \quad = \quad & \int \cos^n x \, dx, \ \int \sin^n x \, dx, \ \int x^n \cos x \, dx, \ \int x^n \sin x \, dx, \\
& \int \sec^n x \, dx, \ \int \csc^n x \, dx, \ \int x^n e^x \, dx, \ \int (\ln x)^n \, dx, \\
& \int e^x \cos^n x \, dx, \ \int e^x \sin^n x \, dx, \ \int \frac{dx}{(x^2 + a^2)^n}, \ \int \frac{dx}{(a^2 - x^2)^n},
\end{aligned}
$$

we may use integration by parts to find a formula to express I_n in terms of I_k with $k < n$. Such a formula is called reduction formula.

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Let

$$I_n = \int x^n \cos x \, dx$$

for positive integer n. Prove that

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}, \text{ for } n \geq 2$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof.

$$
\begin{aligned}
I_n = \int x^n \cos x \, dx &= \int x^n d\sin x \\
&= x^n \sin x - \int \sin x \, dx^n \\
&= x^n \sin x - n \int x^{n-1} \sin x \, dx \\
&= x^n \sin x + n \int x^{n-1} d\cos x \\
&= x^n \sin x + n x^{n-1} \cos x - n \int \cos x \, dx^{n-1} \\
&= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx \\
&= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}
\end{aligned}
$$

$\square$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Let
$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$
where $a > 0$ is a positive real number for positive integer n. Prove that

$$I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, \text{ for } n \geq 2$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

$$
\begin{aligned}
I_n = \int \frac{dx}{(x^2 + a^2)^n} &= \frac{x}{(x^2 + a^2)^n} - \int x\, d\left(\frac{1}{(x^2 + a^2)^n}\right) \\
&= \frac{x}{(x^2 + a^2)^n} + \int \frac{2nx^2 dx}{(x^2 + a^2)^{n+1}} \\
&= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2 - a^2) dx}{(x^2 + a^2)^{n+1}} \\
&= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{dx}{(x^2 + a^2)^n} - 2na^2 \int \frac{dx}{(x^2 + a^2)^{n+1}} \\
&= \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2 I_{n+1} \\
I_{n+1} &= \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n
\end{aligned}
$$

Replacing n by $n-1$, we have

$$
I_n = \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}.
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Alternative proof.

$$
\begin{aligned}
I_n &= \frac{1}{a^2} \int \frac{x^2 + a^2 - x^2}{(x^2 + a^2)^n} dx \\
&= \frac{1}{a^2} \int \left(\frac{1}{(x^2 + a^2)^{n-1}} - \frac{x^2}{(x^2 + a^2)^n} \right) dx \\
&= \frac{1}{a^2} I_{n-1} - \frac{1}{2a^2} \int \frac{x}{(x^2 + a^2)^n} d(x^2 + a^2) \\
&= \frac{1}{a^2} I_{n-1} + \frac{1}{2(n-1)a^2} \int x \, d\left(\frac{1}{(x^2 + a^2)^{n-1}} \right) \\
&= \frac{1}{a^2} I_{n-1} + \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} - \frac{1}{2(n-1)a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}} \\
&= \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \left(\frac{1}{a^2} - \frac{1}{2(n-1)a^2} \right) I_{n-1} \\
&= \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}
\end{aligned}
$$

$\square$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Prove the following reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

for $n \geq 2$. Hence show that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \begin{cases} \dfrac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3} & \text{when } n \text{ is odd} \\ \dfrac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\cdot\dfrac{\pi}{2} & \text{when } n \text{ is even} \end{cases}$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

$$\int \sin^n x \, dx = -\int \sin^{n-1} x \, d\cos x$$

$$= -\cos x \sin^{n-1} x + \int \cos x \, d\sin^{n-1} x$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x \, dx$$

$$n \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

Hence when n is odd

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n x\,dx &= -\left[\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx \\
&= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx \\
&= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\frac{\pi}{2}} \sin^{n-4} x\,dx \\
&\vdots \\
&= \frac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3}\int_0^{\frac{\pi}{2}} \sin x\,dx \\
&= \frac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3}
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof.

when n is even

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n x \, dx &= -\left[\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx \\
&= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx \\
&= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\frac{\pi}{2}} \sin^{n-4} x \, dx \\
&\vdots \\
&= \frac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\int_0^{\frac{\pi}{2}} dx \\
&= \frac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}
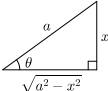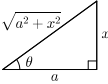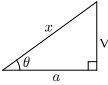\end{aligned}
$$

□

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

$$I_n = \int x^n e^x dx; \qquad I_n = x^n e^x - n I_{n-1}, \ n \geq 1$$

$$I_n = \int (\ln x)^n dx; \qquad I_n = x(\ln x)^n - n I_{n-1}, \ n \geq 1$$

$$I_n = \int x^n \sin x dx; \qquad I_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}, \ n \geq 2$$

$$I_n = \int \cos^n x dx; \qquad I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}, \ n \geq 2$$

$$I_n = \int \sec^n x dx; \qquad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}, \ n \geq 2$$

$$I_n = \int e^x \cos^n x dx; \qquad I_n = \frac{e^x \cos^{n-1} x(\cos x + n \sin x)}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2}, \ n \geq 2$$

$$I_n = \int e^x \sin^n x dx; \qquad I_n = \frac{e^x \sin^{n-1} x(\sin x - n \cos x)}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2}, \ n \geq 2$$

$$I_n = \int x^n \sqrt{x+a} dx; \qquad I_n = \frac{2x^n(x+a)^{\frac{3}{2}}}{2n+3} - \frac{2na}{2n+3} I_{n-1}, \ n \geq 1$$

$$I_n = \int \frac{x^n}{\sqrt{x+a}} dx; \qquad I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2na}{2n+1} I_{n-1}, \ n \geq 1$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Trigonometric substitution

Techniques (Trigonometric substitution)

Expression	Substitution	dx	Trigonometric ratios
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$dx = a\cos\theta \, d\theta$	$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a}$ $\quad\sin\theta = \dfrac{x}{a}$ $\quad\tan\theta = \dfrac{x}{\sqrt{a^2-x^2}}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta \, d\theta$	$\cos\theta = \dfrac{a}{\sqrt{a^2+x^2}}$ $\quad\sin\theta = \dfrac{x}{\sqrt{a^2+x^2}}$ $\quad\tan\theta = \dfrac{x}{a}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta \, d\theta$	$\cos\theta = \dfrac{a}{x}$ $\quad\sin\theta = \dfrac{\sqrt{x^2-a^2}}{x}$ $\quad\tan\theta = \dfrac{\sqrt{x^2-a^2}}{a}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

2. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

3. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \cos^{-1} \left| \frac{a}{x} \right| + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

1. *Let* $x = a \sin \theta$. *Then*

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta \\
dx &= a \cos \theta \, d\theta
\end{aligned}$$

Therefore

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx &= \int \frac{1}{a \cos \theta} (a \cos \theta \, d\theta) \\
&= \int d\theta \\
&= \theta + C \\
&= \sin^{-1} \frac{x}{a} + C
\end{aligned}$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

2. *Let $x = a\tan\theta$. Then*

$$
\begin{aligned}
a^2 + x^2 &= a^2 + a^2\tan^2\theta = a^2\sec^2\theta \\
dx &= a\sec^2\theta d\theta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int \frac{1}{a^2 + x^2}\,dx &= \int \frac{1}{a^2\sec^2\theta}(a\sec^2\theta d\theta) \\
&= \frac{1}{a}\int d\theta \\
&= \frac{\theta}{a} + C \\
&= \frac{1}{a}\tan^{-1}\frac{x}{a} + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof.

3. Let's assume a and x are positive and let $x = a\sec\theta$. Then

$$
\begin{aligned}
x\sqrt{x^2 - a^2} &= a\sec\theta\sqrt{a^2\sec^2\theta - a^2} = a^2\sec\theta\tan\theta \\
dx &= a\sec\theta\tan\theta\, d\theta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx &= \int \frac{1}{a^2\sec\theta\tan\theta}(a\sec\theta\tan\theta\, d\theta) \\
&= \frac{1}{a}\int d\theta \\
&= \frac{\theta}{a} + C \\
&= \frac{1}{a}\cos^{-1}\frac{a}{x} + C
\end{aligned}
$$

Note that $\theta = \cos^{-1}\dfrac{a}{x}$ since $\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{a}{x}$.

$\square$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Use trigonometric substitution to evaluate the following integrals.

1. $\displaystyle \int \sqrt{1-x^2}\,dx$

2. $\displaystyle \int \frac{1}{\sqrt{1+x^2}}\,dx$

3. $\displaystyle \int \frac{x^3}{\sqrt{4-x^2}}\,dx$

4. $\displaystyle \int \frac{1}{(9+x^2)^2}\,dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. *Let $x = \sin\theta$. Then*

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$
$$dx = \cos\theta d\theta.$$

Therefore

$$\int \sqrt{1-x^2}\, dx = \int \cos^2\theta d\theta$$

$$= \int \frac{\cos 2\theta + 1}{2} d\theta$$

$$= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C$$

$$= \frac{\sin\theta\cos\theta}{2} + \frac{\sin^{-1}x}{2} + C$$

$$= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

2. *Let $x = \tan\theta$. Then*

$$
\begin{aligned}
1 + x^2 &= 1 + \tan^2\theta = \sec^2\theta \\
dx &= \sec^2\theta\, d\theta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int \frac{1}{\sqrt{1+x^2}}\, dx &= \int \frac{1}{\sec x}(\sec^2\theta\, d\theta) \\
&= \int \sec\theta\, d\theta \\
&= \ln|\tan\theta + \sec\theta| + C \\
&= \ln(x + \sqrt{1+x^2}) + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

3. *Let $x = 2\sin\theta$. Then*

$$
\begin{aligned}
\sqrt{4-x^2} &= \sqrt{4-4\sin^2\theta} = 2\cos\theta \\
dx &= 2\cos\theta d\theta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{8\sin^3\theta}{2\cos\theta}(2\cos\theta d\theta) \\
&= 8\int \sin^3\theta d\theta \\
&= -8\int (1-\cos^2\theta)d\cos\theta \\
&= 8\left(\frac{\cos^3\theta}{3} - \cos\theta\right) + C \\
&= \frac{(4-x^2)^{\frac{3}{2}}}{3} - 4(4-x^2)^{\frac{1}{2}} + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

4. *Let $x = 3\tan\theta$. Then*

$$
\begin{aligned}
9 + x^2 &= 9 + 9\tan^2\theta = 9\sec^2\theta \\
dx &= 3\sec^2\theta d\theta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int \frac{1}{(9 + x^2)^2}\, dx &= \int \frac{1}{81\sec^4\theta}(3\sec^2\theta d\theta) = \frac{1}{27}\int \cos^2\theta d\theta \\
&= \frac{1}{54}\int (\cos 2\theta + 1)d\theta = \frac{1}{54}\left(\frac{\sin 2\theta}{2} + \theta\right) + C \\
&= \frac{1}{54}(\cos\theta\sin\theta + \theta) + C \\
&= \frac{1}{54}\left(\frac{3}{\sqrt{9 + x^2}} \cdot \frac{x}{\sqrt{9 + x^2}} + \tan^{-1}\frac{x}{3}\right) + C \\
&= \frac{x}{18(9 + x^2)} + \frac{1}{54}\tan^{-1}\frac{x}{3} + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Integration of rational functions

Definition (Rational functions)

A rational function is a function of the form

$$R(x) = \frac{f(x)}{g(x)}$$

where $f(x), g(x)$ are polynomials with real coefficients with $g(x) \neq 0$.

Techniques

We can integrate a rational function $R(x)$ with the following two steps.

1. *Find the partial fraction decomposition of $R(x)$, that is, express*

$$R(x) = q(x) + \sum \frac{A}{(x-\alpha)^k} + \sum \frac{B(x+a)}{((x+a)^2 + b^2)^k} + \sum \frac{C}{((x+a)^2 + b^2)^k}$$

   where $q(x)$ is a polynomial, A, B, C, α, a, b represent real numbers and k represents positive integer.

2. *Integrate the partial fraction.*

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

Let $R(x) = \dfrac{f(x)}{g(x)}$ be a rational function. We may assume that the leading coefficient of $g(x)$ is 1.

1 (Division algorithm for polynomials) There exists polynomials $q(x), r(x)$ with $\deg(r(x)) < \deg(g(x))$ or $r(x) = 0$ such that

$$R(x) = q(x) + \frac{r(x)}{g(x)}.$$

$q(x)$ and $r(x)$ are the quotient and remainder of the division $f(x)$ by $g(x)$.

2 (Fundamental theorem of algebra for real polynomials) $g(x)$ can be written as a product of linear or quadratic polynomials. More precisely, there exists real numbers $\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n, b_1, \ldots, b_n$ and positive integers $k_1, \ldots, k_m, l_1, \ldots, l_n$ such that

$$g(x) = (x - \alpha_1)^{k_1} \cdots (x - \alpha_k)^{k_m} ((x + a_1)^2 + b_1^2)^{l_1} \cdots ((x + a_n)^2 + b)_n^2)^{l_n}.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Techniques

Partial fractions can be integrated using the formulas below.

- $\displaystyle \int \frac{dx}{(x-\alpha)^k} = \begin{cases} \ln|x-\alpha| + C, & \text{if } k = 1 \\ -\dfrac{1}{(k-1)(x-\alpha)^{k-1}} + C, & \text{if } k > 1 \end{cases}$

- $\displaystyle \int \frac{x\,dx}{(x^2+a^2)^k} = \begin{cases} \dfrac{1}{2}\ln(x^2+a^2) + C, & \text{if } k = 1 \\ -\dfrac{1}{2(k-1)(x^2+a^2)^{k-1}} + C, & \text{if } k > 1 \end{cases}$

- $\displaystyle \int \frac{dx}{(x^2+a^2)^k}$

  $= \begin{cases} \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C, & \text{if } k = 1 \\ \dfrac{x}{2a^2(k-1)(x^2+a^2)^{k-1}} + \dfrac{2k-3}{2a^2(k-1)}\displaystyle\int \frac{dx}{(x^2+a^2)^{k-1}}, & \text{if } k > 1 \end{cases}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Theorem

Suppose $\dfrac{f(x)}{g(x)}$ is a rational function such that the degree of $f(x)$ is smaller than the degree of $g(x)$ and $g(x)$ has only simple real roots, i.e.,

$$g(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$$

for distinct real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$ and $a \neq 0$. Then

$$\frac{f(x)}{g(x)} = \frac{f(\alpha_1)}{g'(\alpha_1)(x - \alpha_1)} + \frac{f(\alpha_2)}{g'(\alpha_2)(x - \alpha_2)} + \cdots + \frac{f(\alpha_k)}{g'(\alpha_k)(x - \alpha_k)}$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof

First, observe that

$$g'(x) = \sum_{j=1}^{k} a(x - \alpha_1)(x - \alpha_2) \cdots \widehat{(x - \alpha_j)} \cdots (x - \alpha_k)$$

where $\widehat{(x - \alpha_i)}$ means the factor $x - \alpha_i$ is omitted. Thus we have

$$
\begin{aligned}
g'(\alpha_i) &= \sum_{j=1}^{k} a(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \cdots \widehat{(\alpha_i - \alpha_j)} \cdots (\alpha_i - \alpha_k) \\
&= a(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \cdots \widehat{(\alpha_i - \alpha_i)} \cdots (\alpha_i - \alpha_k)
\end{aligned}
$$

Since $g(x)$ has distinct real zeros, the partial fraction decomposition takes the form

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_k}{x - \alpha_k}.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Proof.

Multiplying both sides by $g(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$, we get

$$f(x) = \sum_{i=1}^{k} A_i a(x - \alpha_1)(x - \alpha_2) \cdots \widehat{(x - \alpha_i)} \cdots (x - \alpha_k)$$

For $i = 1, 2, \cdots, k$, substituting $x = \alpha_i$, we obtain

$$
\begin{aligned}
f(\alpha_i) &= \sum_{j=1}^{k} A_j a(\alpha_j - \alpha_1)(\alpha_j - \alpha_2) \cdots \widehat{(\alpha_j - \alpha_i)} \cdots (\alpha_j - \alpha_k) \\
&= A_i a(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \cdots \widehat{(\alpha_i - \alpha_i)} \cdots (\alpha_i - \alpha_k) \\
&= A_i g'(\alpha_i)
\end{aligned}
$$

and the result follows. $\qquad\square$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Evaluate the following integrals.

1. $\displaystyle \int \frac{x^5 + 2x - 1}{x^3 - x}\,dx$

2. $\displaystyle \int \frac{9x - 2}{2x^3 + 3x^2 - 2x}\,dx$

3. $\displaystyle \int \frac{x^2 - 2}{x(x - 1)^2}\,dx$

4. $\displaystyle \int \frac{x^2}{x^4 - 1}\,dx$

5. $\displaystyle \int \frac{8x^2}{x^4 + 4}\,dx$

6. $\displaystyle \int \frac{2x + 1}{x^4 + 2x^2 + 1}\,dx$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. *By division and factorization* $x^3 - x = x(x-1)(x+1)$, *we obtain the partial fraction decomposition*

$$\frac{x^5 + 4x - 3}{x^3 - x} = x^2 + 1 + \frac{5x - 3}{x^3 - x} = x^2 + 1 + \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}.$$

Multiply both sides by $x(x-1)(x+1)$ *and obtain*

$$5x - 3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$
$$\Rightarrow \quad A = 3, B = 1, C = -4.$$

Therefore

$$\int \frac{x^5 + 4x - 3}{x^3 - x}\, dx = \int \left(x^2 + 1 + \frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1} \right) dx$$
$$= \frac{x^3}{3} + x + 3\ln|x| + \ln|x-1| - 4\ln|x+1| + C.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

2. *By factorization $2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$, we obtain the partial fraction decomposition*

$$\frac{9x-2}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}.$$

Multiply both sides by $x(x+2)(2x-1)$ and obtain

$$9x - 2 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2)$$
$$\Rightarrow \quad A = 1, B = -2, C = 2.$$

Therefore

$$\int \frac{9x-2}{2x^3+3x^2-2x}\, dx$$
$$= \int \left(\frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1} \right) dx$$
$$= \ln|x| - 2\ln|x+2| + \ln|2x-1| + C.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

3. *The partial fraction decomposition is*

$$\frac{x^2 - 2}{x(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x}.$$

Multiply both sides by $x(x-1)^2$ and obtain

$$x^2 - 2 = Ax + Bx(x-1) + C(x-1)^2$$
$$\Rightarrow \quad A = -1, B = 3, C = -2.$$

Therefore

$$\int \frac{x^2 - 2}{x(x-1)^2} \, dx = \int \left(-\frac{1}{(x-1)^2} + \frac{3}{x-1} - \frac{2}{x} \right) dx$$
$$= \frac{1}{x-1} + 3\ln|x-1| - 2\ln|x| + C.$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

4. *The partial fraction decomposition is*

$$\frac{x^2}{x^4 - 1} = \frac{x^2}{(x^2 - 1)(x^2 + 1)}$$

$$= \frac{1}{2}\left(\frac{1}{x^2 - 1} + \frac{1}{x^2 + 1}\right)$$

$$= \frac{1}{2(x - 1)(x + 1)} + \frac{1}{2(x^2 + 1)}$$

$$= \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{2(x^2 + 1)}$$

Therefore

$$\int \frac{x^2 dx}{x^4 - 1} = \int \left(\frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{2(x^2 + 1)}\right) dx$$

$$= \frac{1}{4}\ln|x - 1| - \frac{1}{4}\ln|x + 1| + \frac{1}{2}\tan^{-1}x + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

5. *By factorization* $x^4 + 4 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$,

$$\int \frac{8x^2}{x^4 + 4}\, dx$$

$$= \int \frac{8x^2\, dx}{(x^2 - 2x + 2)(x^2 + 2x + 2)}\, dx$$

$$= \int 2x \left(\frac{4x}{(x^2 - 2x + 2)(x^2 + 2x + 2)} \right) dx$$

$$= \int 2x \left(\frac{1}{x^2 - 2x + 2} - \frac{1}{x^2 + 2x + 2} \right) dx$$

$$= \int \left(\frac{2x}{(x-1)^2 + 1} - \frac{2x}{(x+1)^2 + 1} \right) dx$$

$$= \int \left(\frac{2(x-1)}{(x-1)^2 + 1} + \frac{2}{(x-1)^2 + 1} - \frac{2(x+1)}{(x+1)^2 + 1} + \frac{2}{(x+1)^2 + 1} \right) dx$$

$$= \ln(x^2 - 2x + 2) + 2\tan^{-1}(x - 1) - \ln(x^2 + 2x + 2) + 2\tan^{-1}(x + 1) + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

6.

$$\int \frac{2x+1}{x^4 + 2x^2 + 1} \, dx$$

$$= \int \frac{2x\,dx}{(x^2+1)^2} + \int \frac{dx}{(x^2+1)^2}$$

$$= \int \frac{d(x^2+1)}{(x^2+1)^2} + \int \frac{x^2+1}{(x^2+1)^2} \, dx - \int \frac{x^2\,dx}{(x^2+1)^2}$$

$$= -\frac{1}{x^2+1} + \int \frac{dx}{x^2+1} - \frac{1}{2} \int \frac{x\,d(x^2+1)}{(x^2+1)^2}$$

$$= -\frac{1}{x^2+1} + \tan^{-1} x + \frac{1}{2} \int x\,d\left(\frac{1}{x^2+1}\right)$$

$$= -\frac{1}{x^2+1} + \tan^{-1} x + \frac{1}{2}\left(\frac{x}{x^2+1}\right) - \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$= \frac{x-2}{2(x^2+1)} + \frac{1}{2} \tan^{-1} x + C$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Find the partial fraction decomposition of the following functions.

1. $\dfrac{5x - 3}{x^3 - x}$

2. $\dfrac{9x - 2}{2x^3 + 3x^2 - 2x}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1 For $g(x) = x^3 - x = x(x-1)(x+1)$, $g'(x) = 3x^2 - 1$. Therefore

$$
\begin{aligned}
\frac{5x-3}{x^3-x} &= \frac{-3}{g'(0)x} + \frac{5(1)-3}{g'(1)(x-1)} + \frac{5(-1)-3}{g'(-1)(x+1)} \\
&= \frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1}
\end{aligned}
$$

2 For $g(x) = 2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$, $g'(x) = 6x^2 + 6x - 2$. Therefore

$$
\begin{aligned}
&\frac{9x-2}{2x^3+3x^2-2x} \\
&= \frac{-2}{g'(0)x} + \frac{9(-2)-2}{g'(-2)(x+2)} + \frac{9(\frac{1}{2})-2}{g'(\frac{1}{2})(2x-1)} \\
&= \frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1}
\end{aligned}
$$

$\square$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

t-substitution

Techniques

To evaluate

$$\int R(\cos x, \sin x, \tan x)dx$$

where R is a rational function, we may use t-substitution

$$t = \tan \frac{x}{2}.$$

Then

$$\tan x = \frac{2t}{1 - t^2}; \ \cos x = \frac{1 - t^2}{1 + t^2}; \ \sin x = \frac{2t}{1 + t^2};$$

$$dx = d(2\tan^{-1} t) = \frac{2dt}{1 + t^2}.$$

We have

$$\int R(\cos x, \sin x, \tan x)dx = \int R\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}, \frac{2t}{1 - t^2}\right) \frac{2dt}{1 + t^2}$$

which is an integral of rational function.

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Example

Use t-substitution to evaluate the following integrals.

1. $\displaystyle \int \frac{dx}{1 + \cos x}$

2. $\displaystyle \int \frac{\sin x \, dx}{\cos x + \sin x}$

3. $\displaystyle \int \frac{dx}{1 + \cos x + \sin x}$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

1. Let $t = \tan \dfrac{x}{2}$, $\cos x = \dfrac{1 - t^2}{1 + t^2}$, $dx = \dfrac{2dt}{1 + t^2}$. *We have*

$$
\begin{aligned}
\int \frac{dx}{1 + \cos x} &= \int \left(\frac{1}{1 + \frac{1 - t^2}{1 + t^2}} \right) \frac{2dt}{1 + t^2} = \int dt = t + C = \tan \frac{x}{2} + C \\
&= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} + C = \frac{2 \cos \frac{x}{2} \sin \frac{x}{2}}{2 \cos^2 \frac{x}{2}} + C = \frac{\sin x}{1 + \cos x} + C
\end{aligned}
$$

Alternatively

$$
\begin{aligned}
\int \frac{dx}{1 + \cos x} &= \int \frac{dx}{2 \cos^2 \frac{x}{2}} = \frac{1}{2} \int \sec^2 \frac{x}{2} dx \\
&= \tan \frac{x}{2} + C = \frac{\sin x}{1 + \cos x} + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

2. Let $t = \tan \dfrac{x}{2}$, $\cos x = \dfrac{1 - t^2}{1 + t^2}$, $\sin x = \dfrac{2t}{1 + t^2}$, $dx = \dfrac{2dt}{1 + t^2}$. *We have*

$$
\begin{aligned}
\int \frac{\sin x\, dx}{\cos x + \sin x} &= \int \frac{\frac{2t}{1 + t^2}}{\frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2}} \frac{2dt}{1 + t^2} \\
&= \int \left(\frac{1}{1 + t^2} + \frac{t}{1 + t^2} + \frac{t - 1}{1 + 2t - t^2} \right) dt \\
&= \tan^{-1} t + \frac{1}{2} \ln |1 + t^2| - \frac{1}{2} \ln |1 + 2t - t^2| + C \\
&= \tan^{-1} t - \frac{1}{2} \ln \left| \frac{1 + 2t - t^2}{1 + t^2} \right| + C \\
&= \tan^{-1} t - \frac{1}{2} \ln \left| \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} \right| + C \\
&= \frac{x}{2} - \frac{1}{2} \ln |\cos x + \sin x| + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

Alternatively

$$
\begin{aligned}
\int \frac{\sin x \, dx}{\cos x + \sin x} &= \frac{1}{2} \int \left(1 - \frac{\cos x - \sin x}{\cos x + \sin x} \right) dx \\
&= \frac{x}{2} - \frac{1}{2} \int \frac{d(\sin x + \cos x)}{\cos x + \sin x} \\
&= \frac{x}{2} - \frac{1}{2} \ln |\cos x + \sin x| + C
\end{aligned}
$$

Limits
Differentiation
Integration

Integration
Techniques of Integration
More Techniques of Integration

Solution

3. Let $t = \tan \dfrac{x}{2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $dx = \dfrac{2dt}{1+t^2}$. We have

$$
\begin{aligned}
\int \frac{dx}{1 + \cos x + \sin x} &= \int \frac{\frac{2dt}{1+t^2}}{1 + \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \\
&= \int \frac{dt}{1+t} \\
&= \ln|1+t| + C \\
&= \ln\left|1 + \tan \frac{x}{2}\right| + C \\
&= \ln\left|1 + \frac{\sin x}{1 + \cos x}\right| + C \\
&= \ln\left|\frac{1 + \cos x + \sin x}{1 + \cos x}\right| + C
\end{aligned}
$$