Solution 4

1.* (Montone Convergence Lemmas for measures) Let $m : \widetilde{\mathcal{M}} \to [0, +\infty]$ be a measure, where $\widetilde{\mathcal{M}}$ is an arbitrary σ-algebra. Show that

(a) if $E_n \uparrow E$ (with each $E_n \in \widetilde{\mathcal{M}}\ \forall\, n$), i.e. $E = \bigcup_{n\in\mathbb{N}} E_n$ and $E_n \subseteq E_{n+1}\ \forall\, n$, then $m(E_n) \uparrow m(E)$;

(b) if $E_n \downarrow E$ (with each $E_n \in \widetilde{\mathcal{M}}\ \forall\, n$), i.e. $E = \bigcap_{n\in\mathbb{N}} E_n$ and $E_n \supseteq E_{n+1}\ \forall\, n$, then $m(E_n) \downarrow m(E)$ provided that $m(E_0) < +\infty$ for some n_0. Provide a counter-example if the added condition is dropped.

Solution. (a) Clearly $\{m(E_n)\}_{n=1}^{\infty}$ is increasing. Let $B_1 = E_1$, $B_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then $\{B_n\}$ are pairwise disjoint and $\bigcup_{n=1}^{N} B_n = E_N$ for all N. Hence

$$\begin{aligned} m(E) &= m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty} B_n) \\ &= \sum_{n=1}^{\infty} m(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(B_n) \\ &= \lim_{N\to\infty} m(E_N). \end{aligned}$$

(b) Clearly $\{m(E_n)\}_{n=1}^{\infty}$ is decreasing. Without loss of generality, we may assume that $m(E_1) < +\infty$. Define $A_n = E_1 \setminus E_n$. Then $A_n \subseteq A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = E_1 \setminus (\bigcap_{n=1}^{\infty} E_n) = E_1 \setminus E$. By (a),

$$m(E_1 \setminus E) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m(E_1 \setminus E_n). \tag{1}$$

Since $m(E_1) < +\infty$, we have $m(E), m(E_n) < +\infty$, and hence

$$m(E_1 \setminus E) = m(E_1) - m(E) \quad \text{and} \quad m(E_1 \setminus E_n) = m(E_1) - m(E_n).$$

This together with (1) implies that

$$m(E_1) - m(E) = m(E_1) - \lim_{n\to\infty} m(E_n),$$

and hence $\lim_{n\to\infty} m(E_n) = m(E)$.

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2. Let $\varphi := \sum_{i=1}^{m} a_i \chi_{E_i}$ (each $a_i \in \mathbb{R}$ and $E_i \in \mathcal{M}$) be a "simple function".

(a) Show by MI that range(φ) is a finite set (and so one can list all its non-zero values $b_1, \ldots, b_N$ for some N, unless φ is the zero-function); show that

$$\varphi = \sum_{j=1}^{N} b_j \chi_{\varphi^{-1}(b_j)} \quad \text{where} \quad \varphi^{-1}(b_j) := \{x : \varphi(x) = b_j\}$$

and

$$\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)} \quad \text{where} \quad b_0 = 0.$$

(Both representations can be referred as $\left\{\begin{array}{l}\text{canonical}\\ \text{normal}\\ \text{standard}\end{array}\right.$ representation of φ.)

(b) Define $\displaystyle\int \varphi := \sum_{j=1}^{N} b_j m(\varphi^{-1}(b_j)) = \sum_{j=0}^{N} b_j m(\varphi^{-1}(b_j))$ (with the convention that $0\cdot\infty = 0$) whenever $\varphi \in \mathcal{S}_0$ or $\varphi \in \mathcal{S}^+$ (meaning that the simple function φ is supported by a set of finite measure or $\varphi \geq 0$). Show that if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with each $a_i \in \mathbb{R}$ and $m(E_i) < +\infty$, then $\int \varphi = \sum_{i=1}^{n} a_i m(E_i)$ provided that E_i's are pairwise disjoint (the added condition can be dropped; see Q3 below).

Solution. (a) We prove by induction on m that $\varphi = \sum_{i=1}^{m} a_i \chi_{E_i}$ has $\#(\mathrm{range}(\varphi)) < \infty$. Here $\#(A)$ denotes the cardinality of A.

For $m = 1$, $\varphi = a_1 \chi_{E_1}$, and hence $\#(\mathrm{range}(\varphi)) \leq 2 < \infty$. Suppose the statement is true for $m = k$. For $m = k+1$, let $\psi = \sum_{i=1}^{k} a_i \chi_{E_i}$. By induction assumption, $\mathrm{range}(\psi) = \{b_1, \ldots, b_N\}$ for some $N < \infty$. Since $\varphi = \psi + a_{k+1}\chi_{E_{k+1}}$, we have

$$\mathrm{range}(\varphi) \subseteq \{b_1, \ldots, b_N, b_1 + a_{k+1}, \ldots b_N + a_{k+1}\},$$

so that $\#(\mathrm{range}(\varphi)) \leq 2N < \infty$. Thus the statement is true by MI.

Since $\mathrm{range}(\varphi)$ is a finite set, we can list all its non-zero values $b_1, \ldots, b_N$ for some N, and write $b_0 = 0$. Then clearly

$$\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)}.$$

(b) Let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $a_i \in \mathbb{R}$, E_i pairwise disjoint and $m(E_i) < +\infty$ for each i. Without loss of generality, we may assume that $a_i \neq 0\ \forall\, i$. Let $\varphi = \sum_{j=1}^{N} b_j \chi_{\varphi^{-1}(b_j)}$ be its canonical representation. Then

$$\bigcup_{i : a_i = b_j} E_i = \varphi^{-1}(b_j) \quad \text{for } j = 1, \ldots, N,$$

and hence

$$\begin{aligned}\int \varphi :&= \sum_{j=1}^{N} b_j m(\varphi^{-1}(b_j)) = \sum_{j=1}^{N} b_j \sum_{i : a_i = b_j} m(E_i) \\ &= \sum_{j=1}^{N} \sum_{i : a_i = b_j} a_i m(E_i) = \sum_{i=1}^{n} a_i m(E_i).\end{aligned}$$

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3. Let $\varphi, \psi \in \mathcal{S}_0$ with their canonical presentations $\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)}$ and $\psi = \sum_{k=0}^{M} c_k \chi_{\psi^{-1}(c_k)}$. Show that $\int(\varphi + \psi) = \int \varphi + \int \psi$.

Solution. Let $B_j := \varphi^{-1}(b_j)$ and $C_k := \psi^{-1}(c_k)$. Since $\{C_0, \ldots C_M\}$ is a partition of $\mathbb{R}$, we have $\sum_{k=0}^{M} \chi_{C_k} = 1$. Then

$$\varphi = \sum_j b_j \chi_{B_j} = \sum_j b_j (\sum_k \chi_{C_k} \cdot \chi_{B_j}) = \sum_j \sum_k b_j \chi_{B_j \cap C_k}.$$

Similarly

$$\psi = \sum_k c_k \chi_{C_k} = \sum_k c_k (\sum_j \chi_{B_j} \cdot \chi_{C_k}) = \sum_j \sum_k c_k \chi_{B_j \cap C_k}.$$

Hence $\varphi+\psi=\sum_j\sum_k(b_j+c_k)\chi_{B_j\cap C_k}\in\mathcal{S}_0$. Since $\{B_j\cap C_k\}_{1\le j\le N,1\le k\le M}$ are pairwise disjoint, it follows from Q2 that

$$\begin{aligned}\int(\varphi+\psi)&=\sum_j\sum_k(b_j+c_k)m(B_j\cap C_k)\\&=\sum_j\sum_k b_jm(B_j\cap C_k)+\sum_j\sum_k c_km(B_j\cap C_k)=\int\varphi+\int\psi.\end{aligned}$$

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4.* (3rd: P.70, Q21)

(a) Let D and E be measurable sets and f a function with domain $D\cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.

(b) Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined by $g(x)=f(x)$ for $x\in D$ and $g(x)=0$ for $x\notin D$ is measurable.

Solution. (a) ($\Rightarrow$): Suppose f is measurable. Then $\{x\in D\cup E:f(x)>\alpha\}$ is measurable for any $\alpha\in\mathbb{R}$. Fix $\alpha\in\mathbb{R}$. Then

$$\{x\in D:f\big|_D(x)>\alpha\}=\{x\in D\cup E:f(x)>\alpha\}\cap D$$

which is measurable. Similarly $\{x\in E:f\big|_E(x)>\alpha\}$ is measurable. Since $\alpha\in\mathbb{R}$ is arbitrary, we have that $f\big|_D$ and $f\big|_E$ are measurable.

($\Leftarrow$): It follows immediately from the following equation:

$$\{x\in D\cup E:f(x)>\alpha\}=\{x\in D:f\big|_D(x)>\alpha\}\cup\{x\in E:f\big|_E(x)>\alpha\}.$$

(b) ($\Rightarrow$): Suppose f is measurable. Then

$$\{x:g(x)>\alpha\}=\begin{cases}\{x\in D:f(x)>\alpha\} & \text{if }\alpha\ge 0,\\ \{x\in D:f(x)>\alpha\}\cup D^c & \text{if }\alpha<0,\end{cases}$$

which is measurable in either cases. Hence g is measurable.

($\Leftarrow$): The converse follows immediately from (a) by taking $E=D^c$ and noting that $g\big|_D=f$.

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5.* (3rd: P.71, Q22)

(a) Let f be an extended real-valued function with measurable domain D, and let $D_1=\{x:f(x)=\infty\}$, $D_2=\{x:f(x)=-\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D\setminus(D_1\cup D_2)$ is measurable.

(b) Prove that the product of two measurable extended real-valued function is measurable.

(c) If f and g are measurable extended real-valued functions and α is a fixed number, then $f+g$ is measurable if we define $f+g$ to be α whenever it is of the form $\infty-\infty$ or $-\infty+\infty$.

(d) Let f and g be measurable extended real-valued functions that are finite almost everywhere. Then $f+g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

Solution. (a) ($\Rightarrow$): Suppose f is measurable. Then D_1 and D_2 are measurable as usual. Hence $D \setminus (D_1 \cup D_2)$ is measurable, and so is $f\big|_{D\setminus(D_1\cup D_2)}$ by 4(a).

($\Leftarrow$): Suppose D_1 and D_2 are measurable and $f_{D\setminus(D_1\cup D_2)}$ is measurable. Then $f\big|_{D_1}$ and $f\big|_{D_2}$ are measurable. It then follows from 4(a) that f is measurable.

(b) Let f and g be measurable extended real-valued functions defined on D. Let $D_1 = \{fg = \infty\}$ and $D_2 = \{fg = -\infty\}$. Then

$$D_1 = \{f = \infty, g > 0\} \cup \{f = -\infty, g < 0\} \cup \{f > 0, g = \infty\} \cup \{f < 0, g = -\infty\},$$

which is measurable. Similarly, D_2 is measurable. Let $h = fg\big|_{D\setminus(D_1\cup D_2)}$ and let $\alpha \in \mathbb{R}$. If $\alpha \geq 0$, then

$$\{x : h(x) > \alpha\} = \{x : f\big|_{D\setminus\{f=\pm\infty\}} \cdot g\big|_{D\setminus\{g=\pm\infty\}} > \alpha\},$$

which is measurable; if $\alpha < 0$, then

$$\{x : h(x) > \alpha\} = \{x : f(x) = 0\} \cup \{x : g(x) = 0\} \cup \{x : f\big|_{D\setminus\{f=\pm\infty\}} \cdot g\big|_{D\setminus\{g=\pm\infty\}} > \alpha\},$$

which is also measurable. Hence fg is measurable by (a).

(c) Let f and g are measurable extended real-valued functions and α is a fixed number. Define $f+g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$. Then

$$D_1 := \{f + g = \infty\} = \{f \in \mathbb{R}, g = \infty\} \cup \{f = g = \infty\} = \{f = \infty, g \in \mathbb{R}\}$$

is measurable, and so is $D_2 := \{f + g = -\infty\}$. Let $h = (f+g)\big|_{D\setminus(D_1\cup D_2)}$ and let $\beta \in \mathbb{R}$. If $\beta \geq \alpha$, then

$$\{x : h(x) > \beta\} = \{x : f\big|_{D\setminus\{f=\pm\infty\}} + g\big|_{D\setminus\{g=\pm\infty\}} > \beta\},$$

which is measurable; if $\beta < 0$, then

$$\begin{aligned}\{x : h(x) > \beta\} = \{f = \infty, g = -\infty\} &\cup \{f = -\infty, g = +\infty\} \\ &\cup \{x : f\big|_{D\setminus\{f=\pm\infty\}} + g\big|_{D\setminus\{g=\pm\infty\}} > \beta\},\end{aligned}$$

which is also measurable. Hence $f+g$ is measurable by (a).

(d) Let f and g are measurable extended real-valued functions that are finite a.e. Then the sets D_1, D_2, $\{x : h(x) > \beta\}$ can be written as unions of sets as in (c), possibly with an additional set of measure zero. Thus these sets are measurable and $f+g$ is measurable.

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6.* Let $\varphi \in \mathcal{S}_0$ and $E \in \mathcal{M}$. Define $\int_E \varphi := \int \varphi \cdot \chi_E$. Show that $\varphi \mapsto \int_E \varphi$ is linear:

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi, \quad \forall\, \varphi, \psi \in \mathcal{S}_0 \text{ and } \alpha, \beta \in \mathbb{R}.$$

Solution. Clearly, if $\varphi \in \mathcal{S}_0$ and $E \in \mathcal{M}$, then $\varphi \cdot \chi_E \in \mathcal{S}_0$. Using the result in Q3, we have

$$\int_E (\varphi + \psi) = \int (\varphi \cdot \chi_E + \psi \cdot \chi_E) = \int \varphi \cdot \chi_E + \int \psi \cdot \chi_E = \int_E \varphi + \int_E \psi, \quad \forall\, \varphi, \psi \in \mathcal{S}_0.$$

It remains to show that

$$\int \alpha\varphi = \alpha \int \varphi, \quad \forall\, \varphi \in \mathcal{S}_0 \text{ and } \alpha \in \mathbb{R}.$$

Let $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{S}_0$ with canonical presentations $\varphi = \sum_{j=0}^{N} b_j \chi_{B_j}$, where $B_j = \varphi^{-1}(b_j)$. Then $\alpha\varphi = \sum_{j=0}^{N} \alpha b_j \chi_{B_j}$, and hence, by Q2(b),

$$\int \alpha\varphi = \sum_{j=0}^{N} \alpha b_j m(B_j) = \alpha \sum_{j=0}^{N} b_j m(B_j) = \alpha \int \varphi.$$

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7.* Let $\varphi \in \mathcal{S}_0$. Show that $A \mapsto \int_A \varphi$ is a linear "signed" measure on $\mathcal{M}$.

Solution. It suffices to show that if $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint, measurable sets, then

$$\int_{\bigcup_{n=1}^{\infty} A_n} \varphi = \sum_{n=1}^{\infty} \int_{A_n} \varphi.$$

Let $\varphi \in \mathcal{S}_0$ with canonical presentations $\varphi = \sum_{j=1}^{N} b_j \chi_{B_j}$, where $B_j = \varphi^{-1}(b_j)$. Since $\{A_n\}_{n=1}^{\infty}$ are pairwise disjoint, we have

$$\begin{aligned}
\int_{\bigcup_{n=1}^{\infty} A_n} \varphi &= \int \varphi \cdot \chi_{\bigcup_{n=1}^{\infty} A_n} = \sum_{j=1}^{N} b_j m(B_j \cap \bigcup_{n=1}^{\infty} A_n) \\
&= \sum_{j=1}^{N} b_j \sum_{n=1}^{\infty} m(B_j \cap A_n) = \sum_{n=1}^{\infty} \sum_{j=1}^{N} b_j m(B_j \cap A_n) \\
&= \sum_{n=1}^{\infty} \int \varphi \cdot \chi_{A_n} = \sum_{n=1}^{\infty} \int_{A_n} \varphi.
\end{aligned}$$

Note that we can interchange the summation on the second line since the sum is absolutely convergent:

$$\begin{aligned}
\sum_{j=0}^{N} \sum_{n=1}^{\infty} |b_j m(B_j \cap A_n)| &\le \max_{1 \le j \le N} |b_j| \sum_{j=1}^{N} \sum_{n=1}^{\infty} m(B_j \cap A_n) = \max_{1 \le j \le N} |b_j| m(\bigcup_{j=1}^{N} B_j) \\
&< \infty.
\end{aligned}$$

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