Hw 6(a)

1. (a proof of Tietze extension th). Let $F \subseteq \mathbb{R}$ be closed and $f: F \to \mathbb{R}$ be cts. Then, by the structure theorem for open sets, $\mathbb{R} \setminus F = \bigcup_{n=1}^{\infty} I_n$ with disjoint open intervals $I_n := (a_n, b_n)$. Let $g: \mathbb{R} \to \mathbb{R}$ be an extension of f ($g = f$ on F) such that g is linear on each $[a_n, b_n] \cap \mathbb{R}$ (or, more generally, g is continuous on each of the infinite intervals I_n, and also continuous on each finite intervals $\bar{I}_n = [a_n, b_n]$ such that $g(a_n) \wedge g(b_n) \leq g(x) \leq g(a_n) \vee g(b_n)$ $\forall x \in [a_n, b_n]$).

Show that g is continuous on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$. By symmetry, we need only show that $g(\cdot)$ is a continuous at x_0 from the right-hand side. If $\exists\, b > x_0$ s.t. $(x_0, b) \subseteq F$ (so $x_0 \in F$ as F is closed) or if $\exists\, b > x_0$ s.t. $(x_0, b) \subseteq \mathbb{R} \setminus F$ (so $(x_0, b) \subseteq I_n$ for some n) then it is easy to see the right-continuity of g at x_0). We may therefore assume that $\forall\, b > x_0$ the interval (x_0, b) intersects both $\mathbb{R} \setminus F$ and F (so $x_0 \in F$). Let $\varepsilon > 0$. Since $g|_F$ is cts at x_0, $\exists\, \delta > 0$ s.t.

$$(1) \quad |g(x) - g(x_0)| < \varepsilon \quad \forall\, x \in F \cap [x_0, x_0 + \delta]$$

Take smaller $\delta > 0$ if necessary, we may further assume that $x_0 + \delta \in F$. It remains to show that

$$(2) \quad |g(x) - g(x_0)| < \varepsilon \quad \forall\, x \in (\mathbb{R} \setminus F) \cap (x_0, x_0 + \delta).$$

To do this let $x \in (\mathbb{R} \setminus F) \cap (x_0, x_0 + \delta)$. Then $\exists\, n \in \mathbb{N}$ s.t. $x \in (a_n, b_n)$. Show that $x_0 \leq a_n < b_n \leq x_0 + \delta$ (Hint: $(a_n, b_n) \subseteq \mathbb{R} \setminus F$ and $x_0, x_0 + \delta \in F$) and, by (1), $|g(\cdot) - g(x_0)| < \varepsilon$ at ends a_n, b_n and so on the entire interval $[a_n, b_n]$ (and especially at x).

2. Let $f: E \to [0,+\infty]$ be measurable and $E \in \mathfrak{M}$. For each $n \in \mathbb{N}$ and each $k = 1, 2, \cdots, n2^n$ let

$$A_{n,k} = \left\{ x \in E : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}$$

$$B_n = \{ x \in E : n \leq f(x) \}$$

and let $\varphi_n : E \to \mathbb{R}$ be defined by

$$\varphi_n := \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n \chi_{B_n}, \text{ i.e.}$$

$$\varphi_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \text{ with } k \in \{1, 2, \cdots, n \cdot 2^n\} \\ n & \text{if } n \leq f(x) \end{cases}$$

Show that φ_n is measurable $\forall n$ and $0 \leq \varphi_n \uparrow f$ on E

3. Let $f: E \to \mathbb{R}$ be measurable and $E \in \mathfrak{M}$. Let $\lambda, c \in \mathbb{R}$, $\lambda \neq 0$. Show that $x \mapsto f(\lambda x)$ and $x \mapsto f(x+c)$ are measurable (on where ?).

4.* Let $f \in \mathcal{L}_1[a,b]$. Convention $\int_b^a f := -\int_a^b f$. Show that

(i) $\displaystyle \int_a^b f = \int_{a/\lambda}^{b/\lambda} f(\lambda x)\, dx \quad (\lambda \neq 0)$

(ii) $\displaystyle \int_a^b f = \int_{a-c}^{b-c} f(x+c)\, dx$

(Each in "three steps": $f = \chi_E$, $0 \leq f$ and then general case)
Q2 and the MCT would be helpful for the case when $0 \leq f \in \mathcal{L}_1[a,b]$.

5. Let I and I_n be open intervals such that $\ell(I) \leq 2\ell(I_n)$ and $I \cap I_n \neq \emptyset$. Let $I_n = (c-r, c+r)$ and $J_n = (c-5r, c+5r)$. Show that $I \subseteq J_n$.