

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2050B Mathematical Analysis I
Tutorial 6 (October 18)

The following were discussed in the tutorial this week:

1 Limit of Functions

Definition 1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a **cluster point** of A if given any $\varepsilon > 0$, there exists $x \in A$, $x \neq c$ such that $|x - c| < \varepsilon$.

Remark. 1. A cluster point may or may not be an element of A .

2. Equivalently, c is a cluster point of A if and only if $V_\delta(c) \cap A \setminus \{c\} \neq \emptyset$ for any $\delta > 0$.

3. We denote the set of cluster points of A by A^c .

Example 1. The following are some subsets of \mathbb{R} and their sets of cluster points.

(a) If $A = \mathbb{N}$, then $A^c = \emptyset$

(b) If $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $A^c = \{0\}$

(c) If $A = (1, 2) \cap \mathbb{Q}$, then $A^c = [1, 2]$

Definition 1.2. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A . For a function $f : A \rightarrow \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In this case, the limit is in fact unique and we write

$$\lim_{x \rightarrow c} f = L, \quad \lim_{x \rightarrow c} f(x) = L \quad \text{or} \quad \lim_{A \ni x \rightarrow c} f = L.$$

Example 2. By virtue of ε - δ definition, show that $\lim_{x \rightarrow 2} \frac{x+6}{x^2-2} = 4$.

Solution. Clearly $f(x) := \frac{x+6}{x^2-2}$ has a natural domain $\mathbb{R} \setminus \{\pm\sqrt{2}\}$, which has 2 as a cluster point.

For $x \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$,

$$|f(x) - 4| = \left| \frac{x+6}{x^2-2} - 4 \right| = \frac{|4x^2 - x - 14|}{|x^2-2|} = \frac{|4x+7|}{|x^2-2|} \cdot |x-2|.$$

If $|x-2| < \frac{1}{2}$, then

$$\frac{3}{2} < x < \frac{5}{2} \implies \frac{1}{4} < x^2 - 2 < \frac{17}{4},$$

and

$$|4x + 7| = |4(x - 2) + 15| \leq 4|x - 2| + 15 \leq 20.$$

Let $\varepsilon > 0$ be given. Take $\delta := \min\left\{\frac{\varepsilon}{80}, \frac{1}{2}\right\}$. Now if $0 < |x - 2| < \delta$, then

$$|f(x) - 4| = \frac{|4x + 7|}{|x^2 - 2|} \cdot |x - 2| < 80 \cdot \frac{\varepsilon}{80} = \varepsilon.$$

Theorem 1.1. Let $A \subset \mathbb{R}$, $f : A \rightarrow \mathbb{R}$, $c \in A^c$ and $L \in \mathbb{R}$.

(1) (Sequential Criterion) $\lim_{x \rightarrow c} f = L$ if and only if

$$\lim_n f(x_n) = L \text{ whenever } (x_n) \text{ is a sequence in } A \setminus \{c\} \text{ convergent to } c.$$

(2) (Divergence Criterion) f does not have a limit at c if and only if

there exists a sequence (x_n) in $A \setminus \{c\}$ convergent to c but $(f(x_n))$ does not converge in \mathbb{R} .

Example 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $\lim_{x \rightarrow c} f(x)$ does not exist for every $c \in \mathbb{R}$.

Solution. Let $c \in \mathbb{R}$. In view of the Divergence Criterion, we want to construct a sequence (x_n) such that $x_n \rightarrow c$ but $(f(x_n))$ diverges.

By the density of \mathbb{Q} , for each $n \in \mathbb{N}$, there exists $r_n \in \mathbb{Q}$ such that

$$c < r_n < c + \frac{1}{n}.$$

Similarly, by the density of $\mathbb{R} \setminus \mathbb{Q}$, for each $n \in \mathbb{N}$, there exists $s_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$c < s_n < c + \frac{1}{n}.$$

Now let (x_n) be the sequence given by $x_n = \begin{cases} r_n & \text{if } n \text{ is odd} \\ s_n & \text{if } n \text{ is even.} \end{cases}$

Then clearly, $\lim(x_n) = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$.

However,

$$f(x_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

so that $\lim(f(x_n))$ does not exist. By Divergence Criterion, $\lim_{x \rightarrow c} f(x)$ does not exist. \blacktriangleleft

Example 4. (HW V, Q7) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x + 2 & \text{if } x \in \mathbb{Q} \\ 3x - 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Exactly at what c does $\lim_{x \rightarrow c} f(x)$ exist? And what is the limit then?