THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 5 (October 11)

The following were discussed in the tutorial this week:

1 The Cauchy Criterion

Definition 1.1. A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for any $\varepsilon > 0$ there exists a natural number K such that

$$|x_n - x_m| < \varepsilon \qquad \text{whenever } m, n \geq K.$$

Example 1. Use the definition to determine whether the following sequences are Cauchy.

(a) $x_n := 1 + \dfrac{1}{2^2} + \cdots + \dfrac{1}{n^2}$

(b) $y_n := 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$

Cauchy Convergence Criterion. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Example 2. If $C > 0$, $0 < r < 1$ and the sequence (x_n) satisfies

$$|x_{n+1} - x_n| \leq Cr^n \qquad \text{for all } n \in \mathbb{N}.$$

Show that (x_n) is a Cauchy sequence.

Definition 1.2. A sequence (x_n) of real numbers is said to be **contractive** if there exists a constant r, $0 < r < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq r|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#}$$

The number r is called the **constant** of the contractive sequence.

Remark. Do not confuse (#) with the following condition:

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#\#}$$

For example, $(\sqrt{n})$ satisfies (##) but it is not contractive.

Theorem 1.1. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Example 3. (Sequence of Fibonacci Fractions) Consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Ratio.

Example 4. Let (x_n) be a sequence of real numbers defined by

$$\begin{cases} x_1 = 1, \quad x_2 = 2, \\ x_{n+2} := \dfrac{1}{3}(2x_{n+1} + x_n) \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

Show that (x_n) is convergent and find its limit.

(**Hint:** Note that $x_{n+2} - x_{n+1} = (-\frac{1}{3})(x_{n+1} - x_n)$ and $x_{n+2} - x_1 = \sum_{k=1}^{n+1}(x_{k+1} - x_k)$.)