THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 4 (October 2, 4)

The following were discussed in the tutorial this week:

1 Test Question

Problem 4(i). *Let $a_n = n^{1/n}$. Find the limit of (a_n).*

Solution. Let $b_n = n^{1/n} - 1$. Then clearly $b_n \geq 0$. Furthermore, using Binomial Theorem, we have

$$n = (1 + b_n)^n = \sum_{k=0}^{n} \binom{n}{k} b_n^k \geq \frac{n(n-1)}{2} b_n^2.$$

Thus

$$0 \leq b_n \leq \sqrt{\frac{2}{n-1}} \qquad \text{for } n \geq 2.$$

Since $\lim \sqrt{\dfrac{2}{n-1}} = 0$, it follows from the Squeeze Theorem that $\lim(b_n) = 0$. Hence $\lim(a_n) = 1$. ◀

2 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.1. Let (x_n) be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a **strictly increasing** sequence of natural numbers. Then the sequence (x_{n_k}) is called a subsequence of (x_n).

Subsequence Theorem. *If (x_n) converges, then any subsequence (x_{n_k}) of (x_n) also converges to the same limit.*

Example 1. Let $\ell \in \mathbb{R}$. Show that a sequence (x_n) converges to ℓ if and only if every subsequence of (x_n) has a further subsequence that converges to ℓ

Theorem 2.1. *Let (x_n) be a sequence of real numbers. Then the following are equivalent:*

(i) *(x_n) does not converge to $x \in \mathbb{R}$.*

(ii) *There exists $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$.*

(iii) *There exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.*

The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence*

Example 2. Prove that a bounded divergent sequence has two subsequences converging to different limits.

Example 3. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$. (**Hint:** If n_k has been chosen, find $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $|x_{n_{k+1}}| > k + 1$.)