MATH4050 Real Analysis

~~**Assignment 3**~~ HW3 - 2022

There are ~~8~~ 10 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. * (3rd: P.52, Q51)
(Upper and lower envelopes of a function) Let f be a real-valued function defined on $[a, b]$. We define the *lower envelope* g of f to be the function g defined by

$$g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x),$$

(other notation $\underline{f}(y) = \sup_{\delta>0} f_\delta(y)$ with $f_\delta(y) = \inf_{x \in V_\delta(y)} f(x)$)

and the *upper envelope* h by

$f^\delta(y) := \sup\{f(x) : x \in V_\delta(y)\}$

$$h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x).$$

$= \bar{f}(y) = \inf_{\delta>0} f^\delta(y)$

Prove the following:

a. For each $x \in [a, b]$, $g(x) \le f(x) \le h(x)$, and $g(x) = f(x)$ if and only if f is lower semicontinuous at x, while $g(x) = h(x)$ if and only if f is continuous at x.

b. If f is bounded, the function g is lower semicontinuous, while h is upper semicontinuous.

c. If φ is any lower semicontinuous function such that $\varphi(x) \le f(x)$ for all $x \in [a, b]$, then $\varphi(x) \le g(x)$ for all $x \in [a, b]$.

2. * (3rd: P.53, Q52)
Let f be a lower semicontinuous function defined ~~for all real numbers~~ on $\mathbb{R}$. What can you say about the sets $\{x : f(x) > a\}$, $\{x : f(x) \ge a\}$, $\{x : f(x) < a\}$, $\{x : f(x) \le a\}$, and $\{x : f(x) = a\}$?

3. * (3rd: P.53, Q53; 4th: P.28, Q56)
Let f be a real-valued function defined ~~for all real numbers~~ on $\mathbb{R}$. Prove that the set C of points at which f is continuous is a G_δ. Hint: $C = \bigcap_{\varepsilon>0} C_\varepsilon$, where $C_\varepsilon := \{z : \exists \delta>0 \text{ s.t. } |f(z_1)-f(z_2)| < \varepsilon, \forall z_1, z_2 \in V_\delta(z)\}$
$z \in C_\varepsilon$ with corr. $\delta \Rightarrow V_{\delta/2}(z) \subseteq C_\varepsilon$.

4. * (3rd: P.53, Q54; 4th: P.28, Q57)
Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set C of points where this sequence converges is a $F_{\sigma\delta}$. Hint: $C = \bigcap_{\varepsilon>0} C_\varepsilon$ with C_ε defined by
$C_\varepsilon := \{z : \exists N \in \mathbb{N} \text{ s.t. } |f_n(z)-f_m(z)| \le \varepsilon\ \forall m,n \ge N\} = \bigcup_{N\in\mathbb{N}} C_{\varepsilon,N}$ with
$C_{\varepsilon,N} := \{z : |\cdots| \le \varepsilon\ \forall m,n \ge N\}$

For Question 5-7, let m be a countably additive measure defined for all sets in a σ-algebra $\mathfrak{M}$ $\mathcal{A}$ for Q5-Q7. Prove that:

In an abstract set: need not be in $\mathbb{R}$

5. (3rd: P.55, Q1; 4th: P.31, Q1)
If A and B are two sets in $\mathfrak{M}$ $\mathcal{A}$ with $A \subset B$, then $m(A) \le m(B)$. This property is called monotonicity.

6. (3rd: P.55, Q2; 4th: P.31, Q2)
Let $\{E_n\}$ be any sequence of sets in $\mathfrak{M}$ $\mathcal{A}$. Then $m(\bigcup E_n) \le \sum mE_n$.

7. (3rd: P.55, Q3; 4th: P.31, Q3)
If there is a set A in $\mathfrak{M}$ $\mathcal{A}$ such that $mA < \infty$, then $m\phi = 0$.

$n(E) := \#(E)\ \forall E \subseteq X$

8. (3rd: P.55, Q4; 4th: P.31, Q4). Let X be a set and
Let nE be ∞ for an infinite set E and be equal to the number of elements in E for a finite set. Show that n is a countably additive set function on $\mathcal{P}(X)$ ~~that is~~ translation invariant. ~~and defined for~~ all ~~sets of real numbers~~. This measure is called the **counting measure**.

Moreover, for $X = \mathbb{R}$, $n : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is

9.* Back to $X = \mathbb{R}$ and $\mathfrak{M}$ the σ-alg of all (Lebesgue) measurable subsets of $\mathbb{R}$ & $m = m^*|_{\mathfrak{M}}$. Recalling that outer-measure $m^* : 2^{\mathbb{R}} \to [0, +\infty]$ satisfies

$$m^*(A) = \inf\{m^*(G) : \text{open } G \supseteq A\} \quad \forall A \in 2^{\mathbb{R}},$$

we define the inner measure m_* by

$$m_*(A) = \sup\{m^*(F) : \text{closed } F \subseteq A\} \quad \forall A \in 2^{\mathbb{R}}.$$

Show (by Littlewood's principle) that if $E \in \mathfrak{M}$ then

$$m_*(E) = m^*(E) \quad (\leq +\infty),$$

and (partially converse)

$$(\#) \quad m_*(E) = m^*(E) < +\infty \Rightarrow E \in \mathfrak{M}.$$

Assuming $\exists\, A \subseteq [0,1]$ s.t. $A \notin \mathfrak{M}$, show that the above implication $(\#)$ is not longer valid: $\exists\, E \subseteq \mathbb{R}$ with $m_*(E) = m^*(E) = +\infty$, but $E \notin \mathfrak{M}$.

10.* Show that $\{B \cup Z : B \in \mathcal{B}, Z \in \mathfrak{M}_0\} = \mathfrak{M}$: $E \in \mathfrak{M}$ iff $E = B \cup Z$ for some Borel-set B and set Z with $m^*(Z) = 0$.