Lect24-20180418

Mar 22, Thursday, 2018 3:27 PM

Result of Algebraic Topology

After you have stirred your coffee, one drop of coffee has gone back to its original position.

Brouwer Fixed Point Theorem

Let $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$.

Every continuous $f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ has a fixed point.

That is, $\exists\ x_0 \in \mathbb{D}^n$ such that $f(x_0) = x_0$

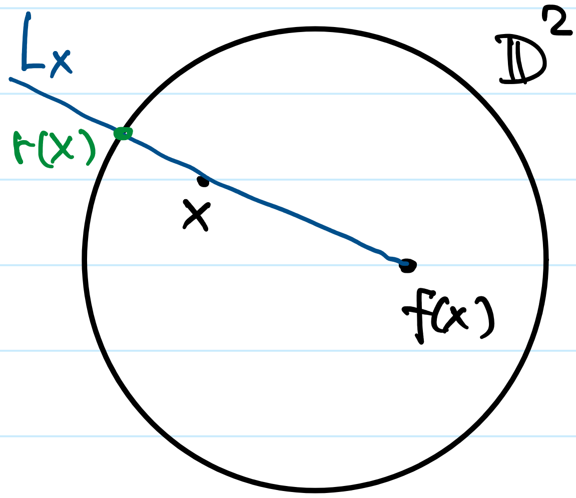
Proof for $n=2$.

Suppose otherwise, $\forall\ x \in \mathbb{D}^2$, $f(x) \neq x$

Then the straight line L_x is defined,

$$L_x = \{(1-t)x + tf(x) : t \in (-\infty, 1]\}$$

And, $L_x \cap S^1$ is a singleton, call it $r(x)$.



In other words,

$$r : \mathbb{D}^2 \longrightarrow S^1$$

is defined and continuous.

Why?

Lect24-p2

Apr 18, Wednesday, 2018 5:05 PM

Indeed, $r(x) = \dfrac{x,\ f(x),\ \langle x, f(x)\rangle,\ \|x\|,\ \|f(x)\|}{\|x - f(x)\|^2}$

solution of a quadratic equation

Moreover, $r|_{S^1} \equiv \mathrm{id}_{S^1}$, i.e., $r \circ i \equiv \mathrm{id}_{S^1} : S^1 \to S^1$

Therefore, $r : D^2 \to S^1$ is a retraction

What's wrong

$$
\begin{tikzcd}
\pi_1(S^1) \arrow[r, "\overset{\text{injective}}{i_\#}"] \arrow[rr, bend left=40, "\mathrm{id}"] \arrow[d, equal] & \pi_1(D^2) \arrow[r, "\overset{\text{surjective}}{r_\#}"] \arrow[d, equal] & \pi_1(S^1) \arrow[d, equal] \\
\mathbb{Z} \arrow[r, bend right=40, "\text{never injective}"'] & 0 \arrow[r, bend right=40, "\text{never surjective}"'] & \mathbb{Z}
\end{tikzcd}
$$

Lect24-p3

Apr 17, Tuesday, 2018 9:22 AM

Another result

There is a place on earth where its temperature and humidity are exactly the same as its "diametrically opposite" place.

can be something else

antipodal

Borsuk-Ulam Theorem

Every continuous $f: S^n \to \mathbb{R}^n$ has $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.

- (temperature, humidity) : $S^2 \longrightarrow \mathbb{R}^2$ is continuous; x_0 and $-x_0$ are antipodal

One Equivalent Version

There is no continuous odd $g: S^n \to S^{n-1}$

$g(-x) = -g(x) \ \forall x$

Lect24-p4

Apr 18, Wednesday, 2018 5:07 PM

"$\Rightarrow$" Regard $g: S^n \to S^{n-1} \subset \mathbb{R}^n$, we have that $x_0 \in S^n$ satisfying

$$g(x_0) = g(-x_0)$$

$$\| \leftarrow g \text{ is odd}$$

$$-g(x_0)$$

$$\therefore g(x_0) = 0 \notin S^{n-1}$$

"$\Leftarrow$ by contrapositive"

Assume $f: S^n \to \mathbb{R}^n$ is continuous, but

$\forall x \in S^n \quad f(x) \neq f(-x)$

Define $g: S^n \longrightarrow S^{n-1}$ by

$$g(x) = \frac{f(x) - f(-x)}{\underbrace{\|f(x) - f(-x)\|}_{\text{non-zero}}}$$

What's wrong?

Therefore, g is continuous

Clearly, $\forall x \in S^n$, $\quad g(-x) = -g(x)$

Lect24-p5

Apr 17, Tuesday, 2018 9:23 AM

Proof of Equivalent version, $n=2$

Let $g: S^2 \longrightarrow S^1$ be continuous.

Assume g is odd, i.e., $\forall\, x \in S^2$, $g(-x) = -g(x)$

Recall that $\mathbb{RP}^n = S^n / (x \sim -x)$

i.e.,
$$\begin{aligned} S^n &\xrightarrow{q_n} \mathbb{RP}^n \\ x &\longmapsto [x] = \{x, -x\} \end{aligned}$$

Then
$$\begin{aligned} \hat{g} : \mathbb{RP}^2 &\longrightarrow \mathbb{RP}^1 \text{ is well-defined} \\ [x] &\longmapsto [g(x)] \\ [-x] &\longmapsto [g(-x)] = [-g(x)] = [g(x)] \end{aligned}$$

Exercise. $\hat{g}$ is continuous (hint: diagram below)

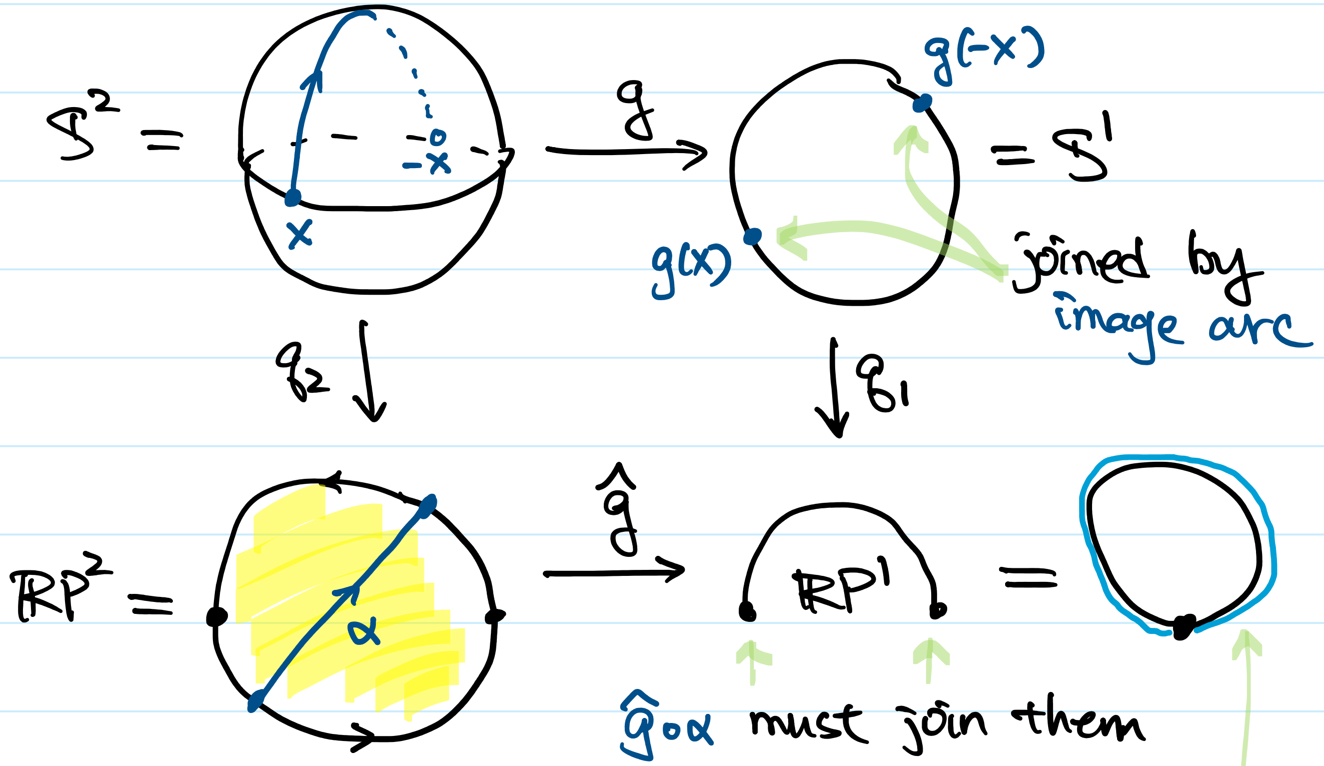
$$\therefore \quad \hat{g}_\# : \pi_1(\mathbb{RP}^2) \longrightarrow \pi_1(\mathbb{RP}^1)$$
$$\pi_1(\mathbb{RP}^2) = (\mathbb{Z}/2, +), \qquad \pi_1(\mathbb{RP}^1) = \,?$$

Exercise. $\mathbb{RP}^1$ is homeomorphic to S^1

So, we have
$$\begin{array}{ccc} \pi_1(\mathbb{RP}^2) & \xrightarrow{\hat{g}_\#} & \pi_1(\mathbb{RP}^1) \\ \| & & \| \\ (\mathbb{Z}/2,+) & \xrightarrow{?} & (\mathbb{Z},+) \end{array}$$

By MATH 2070, $\hat{g}_\# \equiv 0$

Meaning: For every loop α in $\mathbb{RP}^2$,
$\hat{g}\circ\alpha$ is a trivial loop in $\mathbb{RP}^1$



$\hat{g}\circ\alpha$ must join them

cannot be trivial

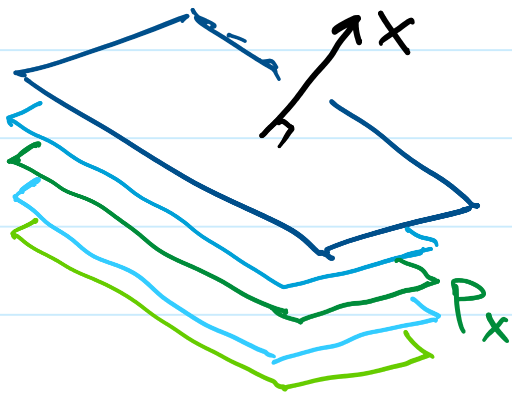
Ham Sandwich Theorem

Given any irregular shape of two pieces of bread and a ham, no matter how they are placed, there is a way to cut them simultaneously into halves.

Proof.

Let $A_1, A_2, A_3 \subset \mathbb{R}^3$ be the two pieces of bread and ham.

Every $x \in S^2$ determines all the cuts (oriented) perpendicular to x.



Choose the one that cuts A_3 into half, and call the plane P_x.

exists by Intermediate Value Theorem

Then P_x also "cuts" each A_1, A_2 into two parts, one in the direction of x, but not necessarily into half.

Let $f_k(x)$ = volume of A_k cut by P_x in the direction of $x \in S^2$

$f = (f_1, f_2) : S^2 \longrightarrow \mathbb{R}^2$ is continuous

By Borsuk-Ulam, $\exists \ x_0 \in S^2$ such that

$$f(x_0) = f(-x_0)$$

i.e., $\underbrace{f_1(x_0) = f_1(-x_0) \quad \text{and} \quad f_2(x_0) = f_2(-x_0)}_{\text{meaning??}}$

$\underbrace{P_{x_0} \text{ and } P_{(-x_0)}}_{\text{same cut but measuring from opposite side}}$ cut A_k into $\underbrace{\text{same size}}_{\text{Equal halves}}$

It already cuts A_3 into equal halves

Q.E.D.